ST. MARY'S SCHOOL

AP Precalculus

Volume 1: Polynomial and Rational Functions

A student reference organized by the College Board Course and Exam Description. Explanations, worked examples, graphs, and brief checks are collected here. Classwork, homework, and substantial response space are in the separate workbook.

CED Unit 1 · Topics 1.1-1.14

Full-color review edition · November 2026

A reference, a place to practice, and a way to explain your reasoning.

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Read, reason, and return

Begin a topic by identifying the quantities and the question. Read the worked examples for both the method and the interpretation. Attempt each brief check before consulting the separate review key. Keep this book for later algebra and Calculus reference.

Four representations, one function

Move among words, tables, graphs, and equations. Use one representation to check another, especially when a calculator display could hide an excluded input or a distant trend.

Color carries local meaning and is paired with labels or line styles. Navy identifies structure; teal highlights ideas; gold marks checks. Graph captions explain which curves and guides to compare.

Topic numbers follow the CED, not a daily lesson schedule. A topic can span several class meetings. Calculator advice is marked in the text; algebraic reasoning and contextual explanations remain essential.

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The College Board AP Precalculus CED effective Fall 2026 sets the coverage. Its topic numbers and learning goals guide this volume. Instructor-manual solution text has not been reproduced. Source sections and original example identifiers are tracked in the editorial records.

This is a full-unit review edition. Review the content, pacing, and a physical duplex/punched copy before a class print run.

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1.1 Change in Tandem

Volume 1 | Polynomial and Rational Functions

A function describes how an output depends on an input. To understand the relationship, we need more than isolated answers: we need to know how the two quantities change together.

In this topic

Interpret functions in words, tables, formulas, and graphs; compare function values; distinguish direction of change from concavity; and sketch a graph from a context.

Functions and their values

A **function** assigns exactly one output to each input in its **domain**. Its **range** is the set of outputs it actually produces. The **input** is the independent variable; the **output** is the dependent variable. Different inputs may produce the same output.

In $y = f(x)$, f names the function, x is the input, and $f(x)$ is the output. The parentheses do not mean multiplication. A statement such as $f(2) = 12$ identifies the ordered pair $(2, 12)$ on the graph.

EXAMPLE 1.1.1 | Read a function in context

A hypothetical café card begins with \$18. Each drink uses \$3 of the balance. Let n be the number of drinks purchased and $C(n)$ the remaining balance, in dollars. No money is added to the card.

$$C(n) = 18 - 3n$$

Input: drinks n	0	1	2	3	4	5	6
Output: $C(n)$(\$)	18	15	12	9	6	3	0

Solution. $C(2) = 18 - 3(2) = 12$ means that \$12 remains after two drinks. The domain is {0, 1, 2, 3, 4, 5, 6}; the range is {0, 3, 6, 9, 12, 15, 18}. The graph consists of seven separate points because a number of whole drinks is a discrete input.

Each additional drink lowers the balance by \$3. The balance is decreasing as the number of drinks increases. Substituting 7 into the expression gives -3, but $C(7)$ is undefined because 7 is outside the domain.

CHECK 1.1.C1

For the café card, interpret $C(4)$. Is 4 an input or an output? Explain why $C(2.5)$ is outside the model even though the formula can be evaluated.

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Images and preimages

The **image** of an input is its single output. The **preimage** of an output is the set of all inputs that produce it. Finding an image means evaluating a function. Finding a preimage means solving for inputs. A preimage may contain several inputs, one input, or no inputs.

EXAMPLE 1.1.2 | Evaluate, then solve

Let $h(p) = p^2 + 2p$, with domain all real numbers. Find the image of 4 and the preimage of 3.

$$h(4) = 4^2 + 2(4) = 24$$

Solution. The image of 4 is 24. For the preimage of 3, set the output equal to 3 and solve:

$$\begin{aligned} p^2 + 2p &= 3, \\ p^2 + 2p - 3 &= 0, \\ (p+3)(p-1) &= 0. \end{aligned}$$

Thus $p = -3$ or $p = 1$, so the preimage of 3 is $\{-3, 1\}$. Substitution checks both inputs: $h(-3) = 9 - 6 = 3$ and $h(1) = 1 + 2 = 3$.

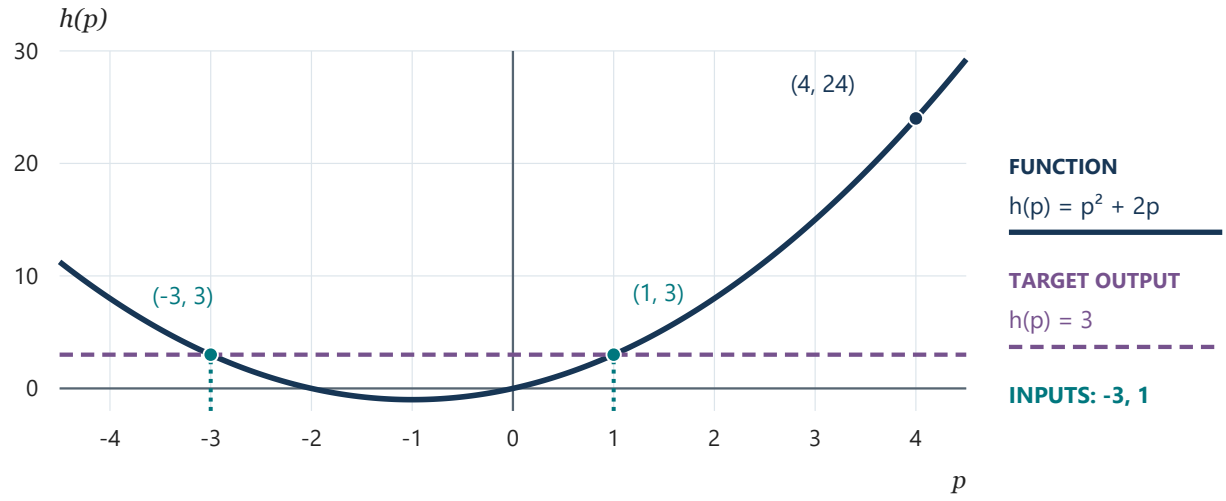


Figure 1.1.1 The dashed output line $h(p) = 3$ meets the graph at two inputs, -3 and 1. Redrawn and adapted from OpenStax section 1.1, Figure 6.

The formula, graph, ordered pairs $(-3, 3)$ and $(1, 3)$, and statement "both inputs give 3" express the same relationship. A vertical line meets a function's graph at most once; a horizontal line may meet it more than once.

CHECK 1.1.C2

For this same function, find the image of -2 and the preimage of 0. Does having two inputs for one output violate the definition of a function?

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A function includes its domain

A formula alone does not completely specify a function. The permitted inputs matter. A context may restrict a formula that otherwise makes sense for many more inputs, as in the café-card example.

Equality of functions

Two functions are equal when they have the same domain and give the same output for every input in that domain. Matching a few values is not enough.

EXAMPLE 1.1.3 | Compare functions, not just formulas

Consider the following three functions.

Function rule	Domain
$f(x) = (x-2)^2$	All real numbers
$g(x) = x^2 - 4x + 4$	All real numbers
$k(x) = (x-2)^2$	$x \geq 2$

Solution. Expanding gives $(x-2)^2 = x^2 - 4x + 4$. Thus f and g have the same domain and identical outputs throughout it: they are equal functions.

The functions f and k use the same formula but have different domains. For example, $f(0) = 4$, while $k(0)$ is undefined. They agree for $x \geq 2$, but they are not equal functions.

Domain restrictions also change preimages. Under f, the preimage of 4 is $\{0, 4\}$. Under k, it is $\{4\}$. A solution of an equation must still be checked against the function's domain.

A short notation refresher

The interval $[a, b]$ includes both endpoints; (a, b) includes neither. A mixed interval such as $[a, b)$ includes only a. For $x \geq a$, write $[a, \infty)$; infinity is not an endpoint value.

Use interval notation for continuous sets of inputs. Use braces for a discrete set, such as $\{0, 1, 2\}$. State a restriction in words or with an inequality when that makes the context clearer.

CHECK 1.1.C3

Let $u(x) = x^2$ on all real numbers and $v(x) = x^2$ on $x \geq 0$. Are they equal? Compare the preimage of 9 under each.

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Increasing, decreasing, and zero

Always compare inputs from smaller to larger. On an interval, a function is **increasing** if $a < b$ always gives $f(a) < f(b)$. It is **decreasing** if $a < b$ always gives $f(a) > f(b)$. A constant function has equal outputs throughout an interval.

EXAMPLE 1.1.4 | Read direction and intercepts

Let $f(x) = (x-2)^2 - 1$ for $0 \le x \le 4$. Describe its change and identify its zeros.

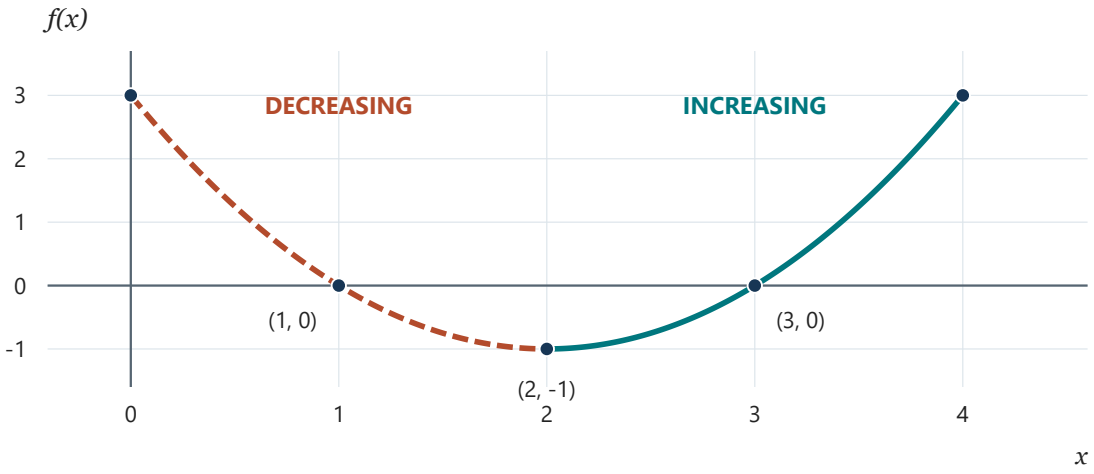


Figure 1.1.2 Direction describes how outputs change; the sign of an output describes its height relative to the horizontal axis.

Solution. From $x = 0$ to 2, the distance from x to 2 gets smaller, so $(x-2)^2$ decreases. From 2 to 4, that distance gets larger, so the square increases. Thus f decreases on $(0, 2)$ and increases on $(2, 4)$. Its smallest output is -1, at $x = 2$.

A **zero** is an input where the output equals zero. Solving $(x-2)^2 - 1 = 0$ gives $(x-2)^2 = 1$, so $x = 1$ or 3. The corresponding intercepts are the points $(1, 0)$ and $(3, 0)$. A graph may touch the horizontal axis at a zero without crossing it.

Keep these claims separate

On $(0, 1)$, f is positive and decreasing. On $(2, 3)$, f is negative and increasing. "Positive" does not mean "increasing."

We use open intervals to report direction consistently. AP Precalculus does not distinguish open from closed endpoints in answers about increasing or decreasing intervals. Domain endpoints still matter when stating the domain.

CHECK 1.1.C4

For f above, describe both its sign and its direction on $(3, 4)$. State its range.

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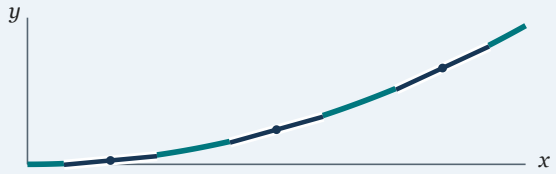
Concavity and changing rates

Direction tells us whether outputs rise or fall. **Concavity** tells us how the rate of change behaves. Think about the slopes of a smooth graph as you move from left to right: positive slopes mean rising outputs, negative slopes mean falling outputs, and greater slope means a greater signed rate.

Concavity

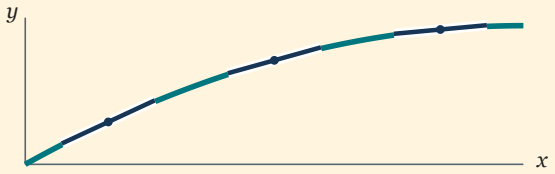
A graph is concave up where its rate of change increases. It is concave down where its rate of change decreases. Either shape can occur while the function itself is increasing or decreasing.

A Increasing **Concave up**



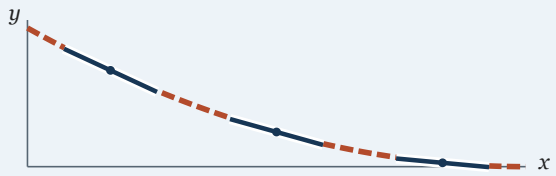
Signed rates: +1, +3, +5

B Increasing **Concave down**



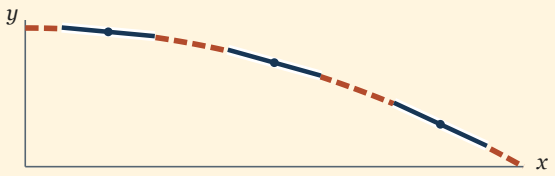
Signed rates: +5, +3, +1

C Decreasing **Concave up**



Signed rates: -5, -3, -1

D Decreasing **Concave down**



Signed rates: -1, -3, -5

Figure 1.1.3 Short navy guides show local slopes; signed rates are listed from left to right. Solid teal curves increase; dashed rust curves decrease.

EXAMPLE 1.1.5 | A decreasing function can be concave up

Suppose the rates of a smooth decreasing function progress from -5 to -3 to -1 as the input increases, and the rate increases throughout the interval.

Solution. The outputs fall because the rates are negative. The graph is concave up because $-5 < -3 < -1$: its signed rate increases. The decline becomes slower in magnitude. In contrast, rates progressing from -1 to -3 to -5 indicate a faster decline and concave-down behavior, if the rate decreases throughout.

CHECK 1.1.C5

A quantity is increasing, but its rate decreases throughout an interval. Which panel matches it? Could its graph be below the horizontal axis?

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Construct a graph from a story

EXAMPLE 1.1.6 | Fill, pause, and drain a tank

Let $H(t)$ be the water depth, in decimeters, t minutes after an empty tank begins filling. During the first two minutes, the depth rises to 4 dm while its rate steadily decreases to zero. It stays at 4 dm until minute 4. Then the depth begins falling, slowly at first and increasingly rapidly, reaching zero at minute 6. Sketch a possible graph.

Solution. Put time on the horizontal axis and depth on the vertical axis. Mark $(0, 0)$, $(2, 4)$, $(4, 4)$, and $(6, 0)$. Use the change descriptions to connect the points.

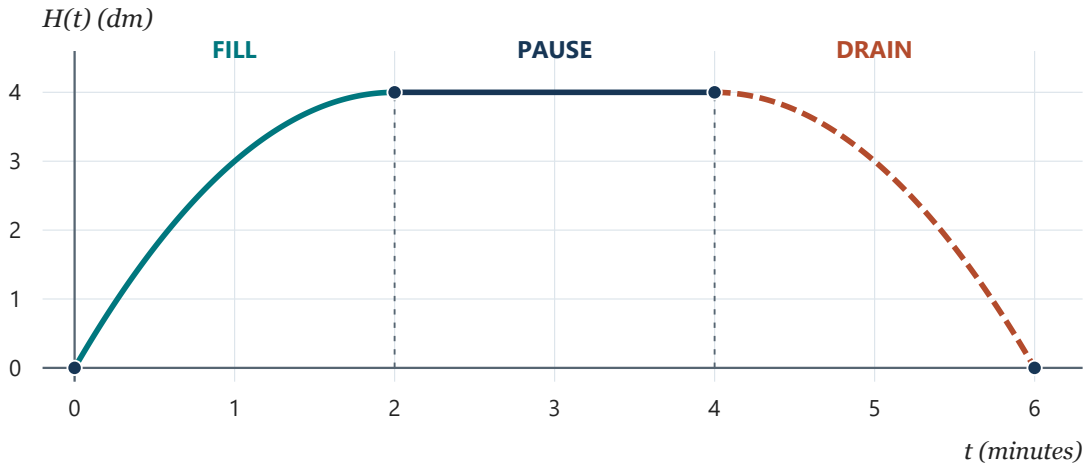


Figure 1.1.4 One possible graph. Colors connect each stage to the table below. The story does not determine a unique formula.

Time interval	Depth	Rate and shape
$0 < t < 2$	Increasing	Positive rate decreases; concave down
$2 < t < 4$	Constant	Zero rate; horizontal segment
$4 < t < 6$	Decreasing	Negative rate decreases; concave down

The modeled domain is $[0, 6]$ minutes and the range is $[0, 4]$ dm. The zeros are $t = 0$ and $t = 6$, when the tank is empty. A horizontal segment means unchanged depth; it does not by itself tell us whether water flows in and out at equal rates.

CHECK 1.1.C6

Change the final part of the story: the depth falls rapidly at first and then more slowly. Describe the new direction and concavity. Which part of your sketch must change?

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What a representation can tell you

A formula can establish an exact relationship over its stated domain. A table may show only selected values. A graph may show only a viewing window. Keep the strength of your conclusion consistent with the information provided.

EXAMPLE 1.1.7 | Avoid claiming more than a table shows

A table lists selected values of a continuous function F.

Input x	0	1	2
Output $F(x)$	0	1	4

Can you conclude that F is increasing and concave up throughout $[0, 2]$?

Solution. The listed outputs increase, and the changes for equal input steps are $+1$ and $+3$. These values are consistent with the increasing, concave-up function $F(x) = x^2$ on $[0, 2]$. They do not prove that this is the function or determine its behavior between the listed inputs.

A continuous curve could pass through those same three points while rising and falling between them. A conclusion about every input in an interval requires more information, such as a known formula or an explicit description of the behavior.

Use graphing technology to check a claim

Return to Example 1.1.4. Enter $Y_1 = (X - 2)^2 - 1$. Use a window such as $-0.5 \le X \le 4.5$ and $-1.5 \le Y \le 3.5$. Interpret only the model's domain, $0 \le x \le 4$.

1. Use a table to check the points $(0, 3)$, $(1, 0)$, $(2, -1)$, $(3, 0)$, and $(4, 3)$.

2. Use trace or a zero-finding tool near each intercept. The zeros should agree with the exact algebraic values 1 and 3.

3. Compare outputs on each side of $x = 2$. Adjust the window if an important feature is hidden.

A displayed decimal is usually an approximation. The graph supports your interpretation, but the equation and domain justify the exact results. A calculator may draw a curve outside a contextual domain unless you restrict it.

CHECK 1.1.C7

A table gives $G(0) = 5$ and $G(4) = 1$. What can you say for certain about the change from the first listed output to the second? Does it prove that G decreases throughout $[0, 4]$?

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Topic 1.1 reference summary

Idea	What to identify or explain
Function	Exactly one output for each input in the domain.
Domain and range	Permitted inputs and the outputs actually produced.
Image and preimage	Evaluate for an image; solve and check the domain for a preimage.
Equal functions	Same domain and identical outputs at every input in it.
Increasing or decreasing	Compare output values as input values increase.
Concave up or down	Determine whether the signed rate increases or decreases.
Zero and intercept	An input a with $f(a) = 0$; the graph contains $(a, 0)$.

A reliable order for interpreting a function

Define the quantities, units, and domain. Read values and intercepts; describe direction. Then describe how the rate changes. State which conclusions are exact, estimated, or dependent on modeling assumptions.

CHECK 1.1.C8

A function is decreasing and concave down throughout an interval. As the input increases, do its outputs increase or decrease? Does its signed rate increase or decrease? Is the decline speeding up or slowing down in magnitude?

CHECK 1.1.C9

A continuous model gives the number of liters in a tank over time. Its graph meets the time axis at 8 minutes. Interpret that point. Does a zero by itself tell you whether the graph crosses the axis?

Looking ahead to Topic 1.2

We will quantify average rate of change over an interval and compare rates near individual inputs. Keep a function value, an output change, and a rate distinct.

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1.2 Rates of Change

FOCUS Compute and interpret average rates from formulas, graphs, and tables. • Compare point rates using short intervals; distinguish value, change, and rate.

From change to rate

A function value describes a quantity at one input. A change compares two outputs. A rate compares that output change with the corresponding input change. These are different pieces of information. A tank can contain a large amount of water while its water level is changing very slowly.

Let $f(x)$ be an output quantity and let $a < b$ be two inputs in its domain. The net change is $f(b) - f(a)$. The average rate of change is the constant rate that would produce this same net change over the interval.

$$\text{Average rate} = \frac{f(b) - f(a)}{b - a}$$

The numerator and denominator must use the same endpoint order. Rate units are output units per input unit. A rate in liters per minute is not a volume in liters. Rearranging the definition gives net change = average rate × input change.

EXAMPLE 1.2.1 | A tank losing water

A tank contains 86 L at 2 min and 50 L at 8 min. Find and interpret its average rate over this interval.

Solution

The output change is $50 - 86 = -36L$. The elapsed time is $8 - 2 = 6\text{min}$.

The average rate is $-\frac{36}{6} = -6\text{L/min}$. The volume decreased by an average of 6 liters per minute over those six minutes.

This describes the interval as a whole. It does not establish that the water drained steadily or decreased at every moment.

CHECK 1.2.C1

An output rises from 14 kg to 35 kg while time rises from 2 h to 5 h. State the output change and average rate, with units.

Prerequisite refresh

Subtraction of a negative increases the difference: $4 - (-2) = 6$. Keep parentheses around substituted negative values.

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A secant connects two observed points

On a graph, average rate is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$. The line records the same endpoint change as the curve. Its height between the endpoints need not agree with the function.

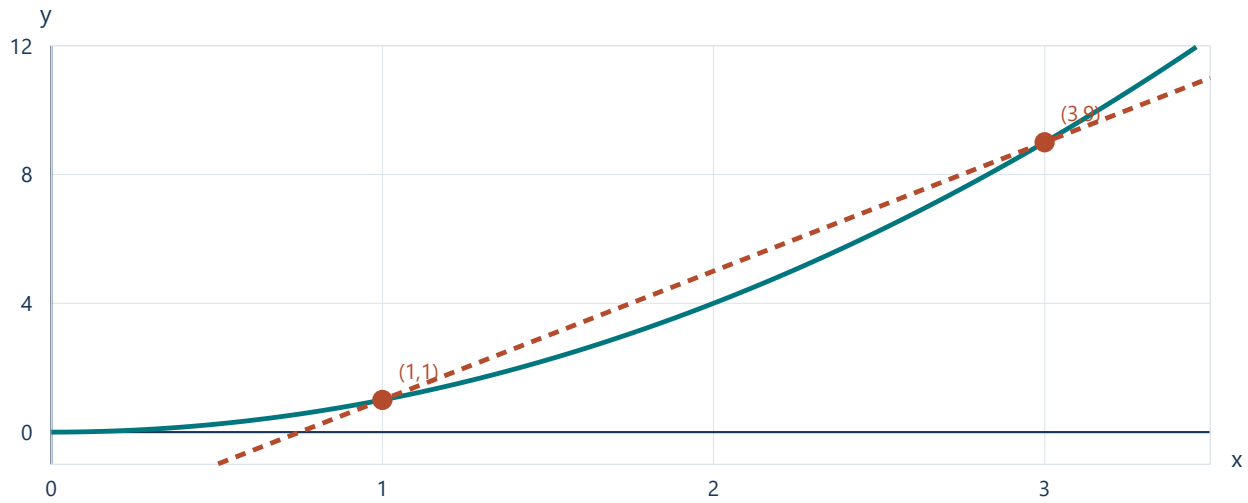


Figure 1.2.1. The solid curve is $f(x) = x^2$; the dashed secant connects $(1, 1)$ and $(3, 9)$.

EXAMPLE 1.2.2 | Rate from a formula

For $f(x) = x^2$, calculate the average rate on $[1, 3]$ and write the secant equation.

Solution

The rate is $\frac{9-1}{3-1} = 4$. A line of slope 4 through $(1, 1)$ has equation $y - 1 = 4(x - 1)$, or $y = 4x - 3$.

At $x = 2$ the curve has output 4 and the secant has output 5. Their agreement at the endpoints does not make them the same function.

CHECK 1.2.C2

Find the average rate of $f(x) = x^2$ on $[-2, 2]$. Does the answer prove that f is constant there?

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Comparing rates near a point

A rate at a point describes how the output would change for a very small change of input near that point. Approximate it with secant slopes over short intervals containing the point. When appropriate nearby slopes settle near the same value, that value estimates the point rate. Derivative rules and formal limit calculations are not needed here.

Interval for $f(x) = x^2$	Output change	Input change	Average rate
$[0.9, 1.1]$	$1.21 - 0.81 = 0.40$	0.20	2
$[0.99, 1.01]$	$1.0201 - 0.9801 = 0.0400$	0.02	2
$[2.9, 3.1]$	$9.61 - 8.41 = 1.20$	0.20	6
$[2.99, 3.01]$	$9.0601 - 8.9401 = 0.1200$	0.02	6

EXAMPLE 1.2.3 | Which point has the larger rate?

Use the table to compare the rates of $f(x) = x^2$ near $x = 1$ and $x = 3$.

Solution

Short-interval slopes near 1 are 2, while those near 3 are 6. The point rate near 3 is larger.

For this quadratic, symmetric intervals happen to produce these same values at every width. That exact agreement is special; nearby rates for a general function usually vary as the interval shrinks.

Use more than one interval width when a graph or table permits it. Rounding, measurement error, or a corner can make estimates unreliable. For $|x|$ at 0, left-hand slopes are -1 and right-hand slopes are 1: a single two-sided point rate is not established.

CHECK 1.2.C3

Nearby rate estimates at A are $-8.1, -8.02$; at B they are $-3.1, -3.02$. Which rate is greater? Where is the output decreasing faster?

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Use the sign and the units

A positive rate means that the quantities change in the same direction: an increase in input corresponds locally to an increase in output. A negative rate means opposite directions. Read a graph from smaller input to larger input before describing increasing or decreasing behavior.

EXAMPLE 1.2.4 | A signed rate in context

Temperature $T(t)$, in °C, is 18 at $t = 0$, 12 at $t = 2$, and 10 at $t = 6$ hours. Compare the average cooling rates.

Solution

On $[0, 2]$, the rate is $\frac{12-18}{2} = -3^\circ\text{C/h}$. On $[2, 6]$, it is $\frac{10-12}{4} = -0.5^\circ\text{C/h}$.

The second rate is greater, but the earlier interval has faster average cooling. The amount lost is also larger earlier: 6 °C versus 2 °C.

The table alone does not prove that the rate changed smoothly or monotonically between observations.

A useful response pattern

State the interval, compute output change divided by input change, include units, and explain the sign. Use the word average when the evidence is an interval rate.

Calculator check: enter the actual function and evaluate both endpoints before dividing. Parenthesize the entire numerator and denominator. A graph window should show both endpoints and the secant; a visually steep graph can be a consequence of unequal axis scales.

CHECK 1.2.C4

A car's s position changes from 12 km to 4 km in 2 h. Find average velocity. Can you determine total distance traveled?

Keep for Calculus

Average rate uses two inputs. A point rate is a local comparison. A zero average rate is compatible with substantial change between the endpoints.

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1.3 Rates of Change in Linear and Quadratic Functions

FOCUS Distinguish output differences from rates and from changes in rates. • Recognize constant linear rates and linearly changing quadratic rates.

Linear functions: the same rate on every interval

For a linear function $L(x) = mx + b$, every secant slope equals m. The starting output b affects values but disappears when two outputs are subtracted. Input spacing need not be one unit and need not be equal from interval to interval.

$$L(v) - L(u) = m(v - u)$$

$$\text{Rate} = \frac{L(v) - L(u)}{v - u} = m, \text{provided} u \neq v$$

EXAMPLE 1.3.1 | Unequal intervals, equal rates

A linear delivery cost is $C(d) = 12 + 1.8d$ dollars for d kilometers. Compare $[0, 5]$, $[5, 12]$, and $[12, 20]$.

Solution

The outputs are 12, 21, 33.6, and 48 dollars. Changes are 9, 12.6, and 14.4 dollars over 5, 7, and 8 km.

Each ratio is 1.8 dollars per kilometer. Unequal output changes do not rule out a linear model when the input intervals differ.

The fixed \$12 charge affects $C(0)$, not the per-kilometer rate.

Quantity	For $L(x) = mx + b$
Output change over an input step h	mh
Average rate over that step	m
Change from one average rate to the next	0

CHECK 1.3.C1

A function takes outputs $7, 13, 25$ at inputs $0, 2, 6$. Are these data consistent with a linear function? Give a formula.

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Quadratic functions and equal input steps

For a quadratic $q(x) = ax^2 + bx + c$ with $a \neq 0$, equal input steps produce output changes that form an arithmetic pattern. Dividing each output change by the input step gives a sequence of average rates that changes by a constant amount.

EXAMPLE 1.3.2 | Two-unit spacing

For $q(x) = 2x^2 - 4x + 1$, compare consecutive intervals from $x = 0$ to $x = 6$ using a step of 2.

Solution

The outputs at $0, 2, 4, 6$ are $1, 1, 17, 49$. First output differences are $0, 16, 32$; second output differences are $16, 16$.

Divide the first differences by 2: the average rates are $0, 8, 16$. Consecutive rates differ by 8.

Those rates are associated with input positions 2 units apart. Their change per input unit is $\frac{8}{2} = 4$. Thus $16, 8$, and 4 describe three different calculations.

Inputs	0	2	4	6
$q(x)$	1	1	17	49
First differences		0	16	32
Average rates		0	8	16

Do not mix these quantities

Second output difference $= 16$. Change in adjacent average rates $= 8$. Rate at which those average rates change $= 4$ per input unit. They coincide only under particular input spacings.

CHECK 1.3.C2

For $q(x) = x^2$ at $x = 0, 3, 6, 9$, find first differences, second differences, and average rates.

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Why the pattern is linear

Let the positive input step be h. Expand $q(x+h)$, subtract $q(x)$, and then divide by h. Terms involving c cancel, as do the terms ax^2 and bx.

$$q(x+h) - q(x) = 2axh + ah^2 + bh$$

$$\text{Average rate} = \frac{q(x+h) - q(x)}{h} = 2ax + ah + b$$

For fixed h, this is a linear function of the interval's left endpoint x. When x increases by h, the average rate increases by $2ah$. Dividing that increase by h gives the constant $2a$. The second output difference is $2ah^2$.

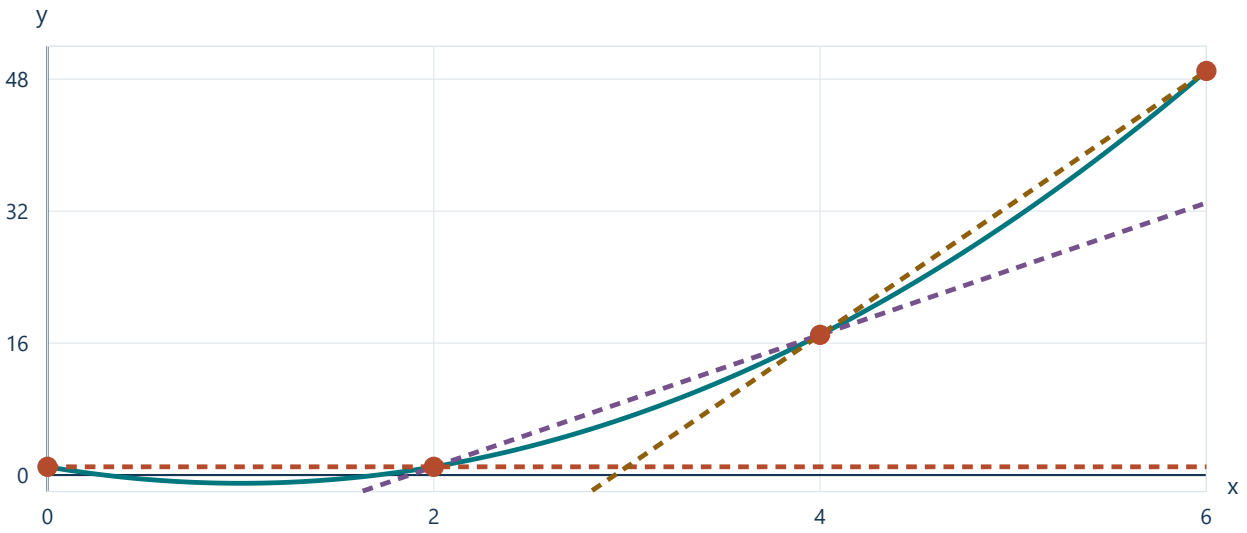


Figure 1.3.1. Equal-width secants for $q(x) = 2x^2 - 4x + 1$ become progressively steeper.

If $a > 0$, the average rates increase and the parabola is concave up. If $a < 0$, the average rates decrease and the parabola is concave down. Concavity describes the direction of change in rates, not whether the outputs are positive.

CHECK 1.3.C3

A quadratic has equal-step average rates $11, 7, 3$ with $h = 2$. Find a and describe concavity.

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Interpret the pattern and its limits

EXAMPLE 1.3.3 | Decreasing, but flattening

A height model has outputs $25, 16, 9, 4, 1$ meters at $t = 0, 1, 2, 3, 4$ seconds. Under a quadratic-model assumption, describe direction and concavity on this observed interval.

Solution

The equal-step average rates are $-9, -7, -5, -3 \text{m/s}$. They are negative, so the sampled heights decrease; the rates increase, so the quadratic is concave up.

The formula $h(t) = (5 - t)^2$ fits every point. On $0 \leq t \leq 4$ the height decreases while its rate becomes less negative.

A sparse table alone cannot guarantee behavior between entries. The stated model assumption supplies that structure.

EXAMPLE 1.3.4 | A sequence and a function

The sequence $a(n) = 3n^2 + 2$ is defined for whole numbers $n \geq 0$. Compare consecutive terms.

Solution

The difference $a(n + 1) - a(n) = 6n + 3$ is linear in n. Thus the differences are $3, 9, 15,\ldots$ and their changes are 6.

The sequence graph consists of discrete points. The same expression on real inputs defines a continuous parabola, a different domain.

Calculator use

Create a table with equally spaced inputs. Record the spacing before computing differences. If spacing is unequal, raw differences do not provide the same degree test.

CHECK 1.3.C4

Two functions agree at five points. One has constant second differences in that table. Must both functions be quadratic everywhere?

Summary: linear outputs have constant average rates. Quadratic outputs have linearly changing average rates. Always record the input spacing and attach the appropriate units.

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1.4 Polynomial Functions and Rates of Change

FOCUS Identify degree, leading coefficient, and extrema. • Connect zeros, turning points, and changes in concavity without derivative rules.

Polynomial structure

A polynomial in x is a finite sum of terms $a_k x^k$ with real coefficients and nonnegative integer exponents. In standard form its nonzero terms are ordered by descending exponent. The largest exponent is the degree; its coefficient is the leading coefficient.

$$p(x) = a_n x^n + \ldots + a_1 x + a_0, a_n \neq 0$$

A nonzero constant has degree 0. The zero polynomial has no degree under the convention used here. An expression with a variable in a denominator, a negative exponent, or a fractional exponent generally is not a polynomial function on its natural domain.

EXAMPLE 1.4.1 | Simplify before classifying

Classify $p(x) = 3x^4 - 2x^2 + x^4 + 7$ and $r(x) = \frac{x^2-1}{x-1}$.

Solution

Combine like terms: $p(x) = 4x^4 - 2x^2 + 7$ has degree 4 and leading coefficient 4.

Although $r(x) = x + 1$ for $x \neq 1$, its original domain excludes 1. It is not equal to the polynomial function $x + 1$ defined on all real numbers. It is a restriction of that polynomial.

An algebraic simplification does not automatically restore a missing domain value.

CHECK 1.4.C1

What are the degree and leading coefficient of $(2x - 1)(x^2 + 3)$?

Vocabulary

A zero is an input. An x-intercept is a point. A maximum or minimum value is an output; also name the input where it occurs.

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Local and absolute extrema

A local maximum output is at least as large as nearby outputs in the domain. A local minimum is at most as large as nearby outputs. An absolute extremum compares with every output in the entire domain. An absolute value may occur at more than one input.

For a nonconstant polynomial, an interior change from increasing to decreasing creates a local maximum; the reverse creates a local minimum. In the AP Precalculus convention, included endpoints of a restricted polynomial domain can also be local extrema, using nearby domain values on the available side.

EXAMPLE 1.4.2 | A restricted parabola

Find local and absolute extrema of $q(x) = (x-2)^2 - 1$ on $[0, 5]$.

Solution

The vertex is $(2, -1)$, where the function changes from decreasing to increasing. It gives the absolute minimum -1 and a local minimum.

At the included endpoints $q(0) = 3$ and $q(5) = 8$. Each is a one-sided local maximum relative to this restricted domain. The absolute maximum is 8 at $x = 5$.

The endpoint $x = 0$ is not an absolute maximum: $q(5) > q(0)$.

EXAMPLE 1.4.3 | Tied absolute minima

For $p(x) = \left(x^2 - 1\right)^2$ on all real numbers, locate its absolute minima.

Solution

A square is never negative, and $p(-1) = p(1) = 0$. Therefore the absolute minimum value is 0 at both $x = -1$ and $x = 1$.

The minimum output is unique as a number, but the input where it occurs need not be unique.

CHECK 1.4.C2

For $f(x) = x^2$ on $[-2, 1]$, give the absolute maximum and minimum values and their locations.

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Turning points and inflection points

A turning point changes the direction of the outputs. An inflection point changes the direction in which the rate is changing: the graph switches between concave up and concave down. These are not the same event.

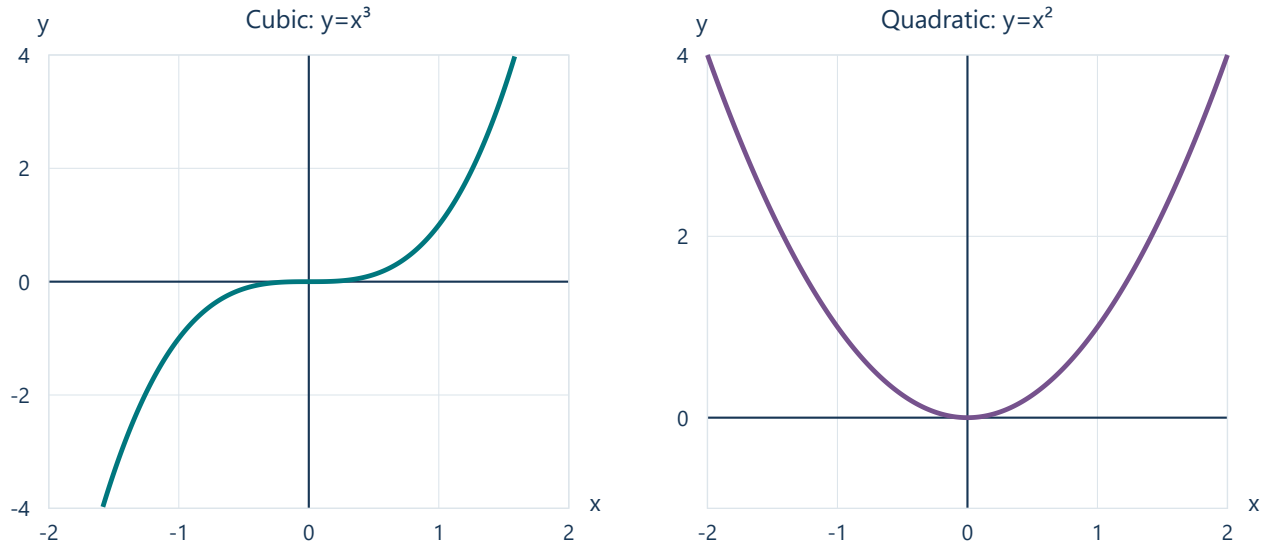


Figure 1.4.1. The cubic x^3 changes concavity at $(0, 0)$ while continuing to increase. The parabola x^2 turns at $(0, 0)$ without changing concavity.

EXAMPLE 1.4.4 | The origin can play different roles

Compare the origin on $f(x) = x^2$ and $g(x) = x^3$.

Solution

For x^2 the outputs decrease to 0 and then increase; it has a local and absolute minimum at the origin. Its rates increase on both sides, so it has no inflection there.

For x^3 the outputs increase through 0. On the negative side the slopes flatten toward 0; on the positive side they steepen away from 0. The graph changes from concave down to concave up, so the origin is an inflection point.

A horizontal-looking point on a graph is not enough to identify a maximum or minimum.

CHECK 1.4.C3

A graph is increasing on both sides of $x = 2$, but changes from concave up to concave down there. What feature does this describe?

Graphing technology can estimate locations. Check a wider window and use nearby tables or slopes to support the interpretation. A small window can hide a turn or make a curved graph appear straight.

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What zeros and degree tell you

A degree$-n$ nonconstant polynomial has at most $n-1$ turning points. Between two distinct real zeros there must be at least one local maximum or minimum. The polynomial must leave the axis and return; it cannot remain identically zero throughout the interval.

EXAMPLE 1.4.5 | Infer a minimum possible degree

A polynomial graph has four distinct real zeros and three turning points. What degree is possible?

Solution

Four distinct real zeros require degree at least 4. Three turning points also require degree at least 4.

Degree 4 is possible, for example a product with four distinct linear factors. The information does not prove the degree is exactly 4; higher-degree polynomials can have the same number of real zeros and turns.

On the full real domain, a nonconstant even-degree polynomial has an absolute minimum if its leading coefficient is positive, or an absolute maximum if the coefficient is negative. It can have several local extrema. Odd-degree polynomials have neither absolute extreme on the full real domain because their two ends go in opposite directions.

Context changes the domain

A polynomial that is unbounded on all real inputs may have absolute extrema on a closed physical interval. Always identify the domain before making a global statement.

CHECK 1.4.C4

Can a cubic have three turning points? Can it have an absolute maximum on a closed restricted interval?

Summary: use algebra for degree, a graph or suitable reasoning for local shape, and the actual domain for absolute comparisons. A feature visible in a window is not automatically a global feature.

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1.5 Polynomial Functions and Complex Zeros

FOCUS Connect factors, zeros, multiplicity, and sign. • Count complex zeros and use conjugate pairs. • Analyze finite differences and even/odd symmetry.

Zeros, roots, and factors

A zero a of p is a number satisfying $p(a) = 0$. It is also a root of the equation $p(x) = 0$. For a real zero a, the factor theorem connects the equation with the factor $x - a$. A real zero gives the graph point $(a, 0)$; a nonreal zero does not give an x-intercept on a real coordinate graph.

EXAMPLE 1.5.1 | Read a useful factorization

Find all zeros of $p(x) = (x + 2)(x - 1)^2(x^2 + 4)$. State their multiplicities.

Solution

The linear factors give -2 once and 1 twice. Solving $x^2 + 4 = 0$ gives $x = 2i$ and $x = -2i$, each once.

The polynomial has degree 5 and five complex zeros counted with multiplicity: $-2, 1, 1, 2i, -2i$. Only $(-2, 0)$ and $(1, 0)$ are x-intercepts.

The real numbers are included among the complex numbers. Five complex zeros does not mean five nonreal zeros.

Complex-number refresher

The imaginary unit i satisfies $i^2 = -1$. A complex number has form $a + bi$ for real a,b. Solve $x^2 = -k$ with $k > 0$ using $x = \pm i\sqrt{k}$; both signs are needed.

CHECK 1.5.C1

List the zeros, with multiplicity, of $(x - 3)^2(x^2 + 9)$. How many real x-intercepts are there?

The Fundamental Theorem of Algebra gives exactly n complex zeros for a degree$-n$ polynomial, counting multiplicity. It gives an upper bound of n distinct real zeros, not a promise that all zeros are real.

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Multiplicity determines nearby sign behavior

If $p(x) = (x - a)^m g(x)$ and $g(a) \neq 0$, then g has a fixed nonzero sign sufficiently near a. The sign change at a is controlled by the factor $(x - a)^m$. Even powers keep the same sign on both sides; odd powers change sign.

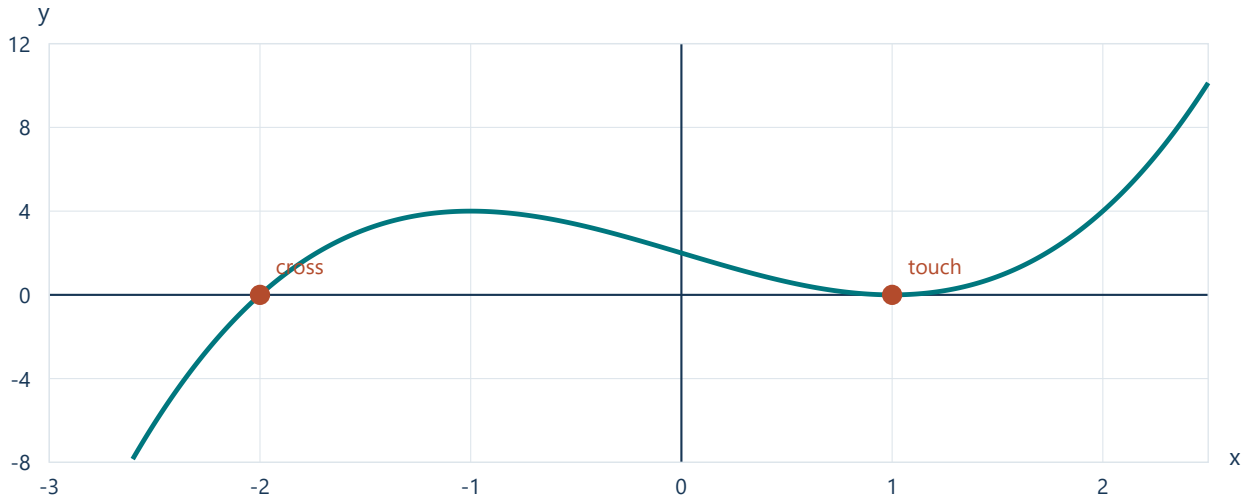


Figure 1.5.1. For $p(x) = (x + 2)(x - 1)^2$, the graph crosses at -2 and touches at 1.

EXAMPLE 1.5.2 | Cross or touch?

Describe the graph near each real zero of $p(x) = (x + 2)(x - 1)^2$.

Solution

At -2 the multiplicity is 1, so the output changes sign and the graph crosses the axis.

At 1 the multiplicity is 2, so nearby nonzero outputs have the same sign. The graph touches the axis and turns.

Odd multiplicities greater than 1 can cross with flattening. The crossing/touching distinction alone does not determine the exact multiplicity.

CHECK 1.5.C2

A graph crosses at $x = 4$ with noticeable flattening. Is multiplicity 2 consistent with that crossing?

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Polynomial inequalities

Between consecutive distinct real zeros, a polynomial has no zero and keeps a constant sign. Factor the polynomial, mark its real zeros on a number line, and test one point in each interval. Include zeros for $\geq$ or $\leq$ and exclude them for strict inequalities.

EXAMPLE 1.5.3 | Solve by signs

Solve $(x+2)(x-1)^2 \leq 0$.

Solution

The critical inputs are -2 and 1. Away from 1, the squared factor is positive, so the sign is the sign of $x+2$.

The expression is negative on $(-\infty, -2)$, positive on $(-2, 1)$ and $(1, \infty)$, and zero at -2 and 1.

The solution is $(-\infty, -2] \cup \{1\}$. The isolated zero at 1 belongs even though there is no surrounding interval of negative outputs.

Interval	$x+2$	$(x-1)^2$	Product
$x < -2$	-	+	-
$-2 < x < 1$	+	+	+
$x > 1$	+	+	+

A sign chart is an algebraic argument. A graph provides a valuable check, but zooming cannot establish whether a curve exactly touches the axis. Use the factorization for exact endpoint inclusion.

CHECK 1.5.C3

Solve $(x+1)(x-2) \geq 0$.

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Conjugates and constructing a polynomial

For a polynomial with real coefficients, nonreal zeros come in conjugate pairs. If $a+bi$ is a zero with $b \neq 0$, then $a-bi$ is also a zero. Multiplying their factors produces a quadratic with real coefficients.

$$(x-(a+bi))(x-(a-bi)) = (x-a)^2 + b^2$$

EXAMPLE 1.5.4 | Use the missing conjugate

Find a real-coefficient polynomial of least degree with zeros 2 and $1+2i$, and with $p(0) = -20$.

Solution

The conjugate $1-2i$ must also be a zero. Their factors give $(x-1)^2 + 4 = x^2 - 2x + 5$.

Write $p(x) = k(x-2)\left(x^2-2x+5\right)$. At $x=0, p(0) = -10k = -20$, so $k=2$.

Thus $p(x) = 2(x-2)\left(x^2-2x+5\right) = 2x^3 - 8x^2 + 18x - 20$. The least possible degree is 3.

Without a condition such as $p(0) = -20$, the zeros determine a family of polynomials differing by a nonzero scale factor. A least-degree requirement prevents adding extra factors with additional zeros.

CHECK 1.5.C4

A real-coefficient quartic has zeros -1 twice and $3+i$. Find its remaining zero.

Optional enrichment

The Rational Zero Theorem and extensive synthetic-division searches can locate useful factors. The core AP task here uses a suitable factorization or technology; do not confuse a list of candidates with verified zeros.

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Finite differences: evidence for degree

For equally spaced inputs, subtract consecutive outputs to make first differences. Repeat the process to make second, third, and higher differences. For a degree $-n$ polynomial with $n \geq 1$, its nth differences are constant and nonzero. Lower-order differences are not constant over the full polynomial pattern.

x	0	1	2	3	4
$p(x) = x^3$	0	1	8	27	64
First differences		1	7	19	37
Second differences			6	12	18
Third differences				6	6

EXAMPLE 1.5.5 | Interpret a difference table

What does the table show about $p(x) = x^3$?

Solution

First differences are not constant, so the function is not linear. Second differences are not constant, so it is not quadratic.

The third differences are constant and nonzero. This matches degree 3.

A single bottom entry in a short table is not evidence of a repeated constant pattern. Also, finite data can be fit by higher-degree functions; infer a plausible least-degree model, not a unique law.

Spacing matters. For step h, the constant nth difference is $n!a_n h^n$. You may use this relationship as a useful extension; the essential requirement is the constant nonzero difference pattern. Do not apply ordinary equal-step differences to irregularly spaced inputs.

CHECK 1.5.C5

Outputs at $x = 0, 1, 2, 3, 4$ are $1, 4, 9, 16, 25$. Under a polynomial-model assumption, identify the least degree and a fitting rule.

A finite-data caution

If q fits inputs $x_1,\ldots,x_k$, then $q(x) + c(x - x_1)\ldots(x - x_k)$ also fits them for any c. Extra assumptions are necessary to select a model.

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Even and odd functions

An even function satisfies $f(-x) = f(x)$, with a domain symmetric about 0. Its graph is symmetric about the vertical axis. An odd function satisfies $f(-x) = -f(x)$; its graph is symmetric about the origin. These properties describe the entire function, not merely its highest power.

EXAMPLE 1.5.6 | Test the whole expression

Classify $f(x) = x^4 - 3x^2 + 2$, $g(x) = x^3 - 2x$, and $h(x) = x^4 + x$.

Solution

For f, replacing x with $-x$ leaves every term unchanged, so f is even.

For g, $g(-x) = -x^3 + 2x = -g(x)$, so g is odd.

For h, $h(-x) = x^4 - x$, which equals neither $h(x)$ nor $-h(x)$ for all x. It is neither even nor odd.

Feature	Even function	Odd function
Algebraic test	$f(-x) = f(x)$	$f(-x) = -f(x)$
Graph symmetry	Vertical axis	Origin
Paired point	(a, b) and $(-a, b)$	(a, b) and $(-a, -b)$

A polynomial containing only even powers, including a constant, is even. One containing only odd powers is odd. The zero function is both. An even degree does not by itself make a function even.

CHECK 1.5.C6

Is $x^3 + 1$ odd? Is the constant function $f(x) = 5$ even?

Keep for Calculus

A factorization tells you where the output is zero and what its sign does nearby. It does not alone list every turning point or inflection point.

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1.6 Polynomial Functions and End Behavior

FOCUS Predict each end using degree and leading coefficient. • Express end behavior in words and limit notation; distinguish local shape from eventual behavior.

The leading term controls the ends

End behavior asks what happens as the input grows without bound in either direction. It does not describe what happens near one particular zero or turning point. Write a polynomial in standard form and identify its leading term $a_n x^n$.

For large $|x|$, the leading term grows in magnitude faster than every lower-degree term. Dividing $p(x)$ by $a_n x^n$ gives 1 plus terms that approach 0. The leading term therefore determines the eventual sign and unbounded direction of p.

EXAMPLE 1.6.1 | A negative leading coefficient

Describe both ends of $p(x) = -2x^5 + 7x^3 - 100x + 4$.

Solution

The leading term is $-2x^5$. For large positive x, x^5 is positive, so $p(x)$ decreases without bound.

For large negative x, x^5 is negative; multiplying by -2 makes the leading term positive. Thus $p(x)$ increases without bound to the left.

Write $\lim_{x \to \infty} p(x) = -\infty$ and $\lim_{x \to -\infty} p(x) = \infty$.

Read the subscript first

$x \to \infty$ means move far to the right. $x \to -\infty$ means move far to the left. The value after the equals sign describes the output. Infinity is a direction of unbounded behavior, not an input you substitute.

CHECK 1.6.C1

Give both end limits of $3x^4 - x + 8$.

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Four cases, with reasons



Figure 1.6.1. Representative power functions display the four possible end patterns. Lower-degree terms can change the middle but not these eventual directions.

Degree	Leading coefficient	Left end	Right end
Even	Positive	$+\infty$	$+\infty$
Even	Negative	$-\infty$	$-\infty$
Odd	Positive	$-\infty$	$+\infty$
Odd	Negative	$+\infty$	$-\infty$

For an even power, both positive and negative inputs give positive powers. For an odd power, the power has the sign of the input. The leading coefficient then preserves or reverses these signs.

CHECK 1.6.C2

A nonconstant polynomial has both ends decreasing without bound. What can you infer about degree and leading coefficient?

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A window is not the entire graph

EXAMPLE 1.6.2 | A small leading coefficient still wins

For $p(x) = 0.01x^4 - x^2$, explain why a small graph window can mislead.

Solution

The leading term is positive and even, so both ends eventually rise without bound.

Yet $p(1) = -0.99$ and $p(5) = -18.75$, so the graph is below the axis at these positive inputs. The quartic and quadratic terms have equal magnitude at $|x| = 10$, where $p(x) = 0$.

At $x = 20, p(20) = 1200$, illustrating the eventual dominance. The word eventually has no fixed universal cutoff.

A useful technology sequence is predict, graph, then widen the input and output windows. If the display seems to contradict a leading-term conclusion, first check the entered expression and the scale. A table at large positive and negative inputs can support the reasoning.

EXAMPLE 1.6.3 | Start with factors

Find the end behavior of $p(x) = -3(x + 2)^2(x - 4)^3$.

Solution

The degrees add to $2 + 3 = 5$. The product of the leading coefficients is -3, so the leading term is $-3x^5$.

Therefore the left end rises and the right end falls. Expanding every term is unnecessary.

CHECK 1.6.C3

Find both end limits of $-4(x - 1)^3(x + 2)$.

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Combine end behavior with other evidence

EXAMPLE 1.6.4 | Constraints on a sketch

A polynomial of degree 4 has positive leading coefficient, crosses at $x = -2$ and $x = 3$, and touches at $x = 1$. Give one possible formula.

Solution

Crossing permits odd multiplicities at -2 and 3; touching permits even multiplicity at 1. The least such multiplicities are $1, 1, 2$ and already total 4.

One choice is $p(x) = (x + 2)(x - 1)^2(x - 3)$. Its leading coefficient is positive, so both ends rise.

A positive constant multiple also works. A graph satisfying only these conditions is not unique.

Before sketching, mark exact zeros and multiplicities, the y-intercept when useful, and both end directions. Connect the pieces smoothly while respecting sign intervals. Technology can refine turning-point locations, but the structural constraints remain the explanation.

Required versus optional

Reading and writing end limits is required. A formal proof using limit laws, detailed asymptotic error bounds, or derivative tests is optional enrichment.

CHECK 1.6.C4

Does $\lim_{x \to \infty} p(x) = \infty$ imply that $p(x) > 0$ for every x?

Summary: determine degree and leading sign; check the two ends separately; write a complete statement for each end. Preserve the distinction between an output value and an unbounded trend.

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1.7 Rational Functions and End Behavior

FOCUS Use leading-term ratios to determine rational end behavior. • Interpret horizontal and slant asymptotes without treating them as barriers.

A quotient compares two polynomials

A rational function has form $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. Its domain excludes every real zero of the original denominator. End behavior compares how quickly the numerator and denominator grow at large positive and negative inputs.

$$r(x) = \frac{p(x)}{q(x)}$$

If the leading terms are ax^m and bx^n, their quotient is $\left(\frac{a}{b}\right)x^{m-n}$. This gives the eventual scale and direction. It does not preserve every intercept, hole, or asymptote location of the original function.

EXAMPLE 1.7.1 | The denominator grows faster

Find the end limits of $r(x) = \frac{3x+2}{x^2+1}$.

Solution

The leading-term ratio is $\frac{3x}{x^2} = \frac{3}{x}$, which approaches 0 from positive values on the right and negative values on the left.

Thus $\lim_{x\to\infty} r(x) = 0$ and $\lim_{x\to-\infty} r(x) = 0$. The horizontal asymptote is $y = 0$.

The denominator $x^2 + 1$ is positive for all real inputs, so this example has no domain exclusions.

CHECK 1.7.C1

Find the horizontal asymptote of $\frac{5x^2-1}{2x^3+7}$, if any.

End behavior is a large-input question

A denominator zero describes behavior near a particular finite input. A horizontal asymptote describes output behavior as the input becomes unbounded.

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Equal degrees and horizontal asymptotes

When numerator and denominator have the same degree, the leading powers cancel. The end limits equal the ratio of the leading coefficients. When the numerator has lower degree, both end limits are 0.

EXAMPLE 1.7.2 | A horizontal asymptote can be crossed

For $r(x) = \frac{2x^2+3x+1}{x^2+1}$, identify its horizontal asymptote and determine whether the graph intersects it.

Solution

Equal degrees give the asymptote $y = 2$.

Solve $r(x) = 2$: $2x^2 + 3x + 1 = 2x^2 + 2$, so $3x = 1$ and $x = \frac{1}{3}$. The denominator is nonzero there.

Thus the graph crosses the asymptote at $\left(\frac{1}{3}, 2\right)$. A horizontal asymptote constrains distant behavior, not all finite-input values.

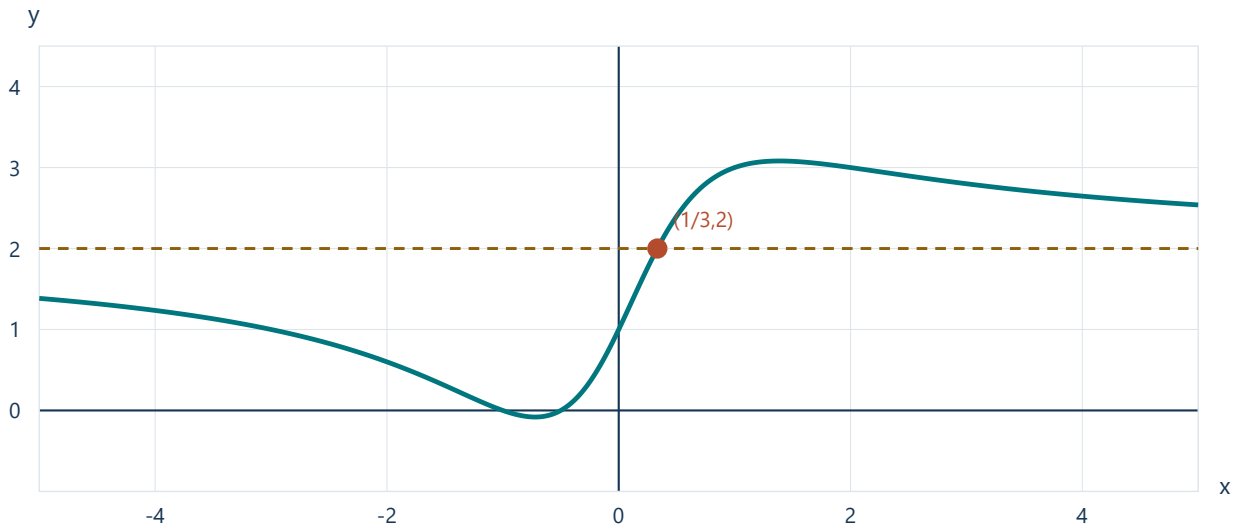


Figure 1.7.1. The solid rational curve crosses the dashed horizontal asymptote $y = 2$.

CHECK 1.7.C2

Find both end limits of $\frac{6x^3-5}{2x^3+x+1}$.

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When the numerator grows faster

If the numerator degree is greater, the quotient of the leading terms is a nonconstant power of x. Its degree and sign determine which way the rational function goes at each end. There is no horizontal asymptote in this case.

EXAMPLE 1.7.3 | A slant slope is not a slant equation

Describe the ends of $r(x) = \frac{2x^2+3x+5}{x+1}$.

Solution

The leading ratio is $2x$, so the right end rises and the left end falls. A slant asymptote, if written as $y = mx + b$, has slope $m = 2$.

Division gives $r(x) = 2x + 1 + \frac{4}{x+1}$, for $x \neq -1$. The remainder term approaches 0 at both ends, so the actual slant asymptote is $y = 2x + 1$.

The leading-term ratio predicts the slope but misses the intercept. Topic 1.11 develops the division method.

EXAMPLE 1.7.4 | Degree difference two

Describe the ends of $\frac{3x^4-x}{x^2+2}$.

Solution

The leading ratio is $3x^2$. Therefore both end limits are $+\infty$.

There is neither a horizontal nor a slant line asymptote. A polynomial asymptote of higher degree is an optional extension, found by division.

CHECK 1.7.C3

Does a leading-term quotient of $-3x$ determine a slant asymptote $y = -3x$ exactly?

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Interpret the distant trend

Compare degrees	Leading-term behavior	Conclusion
$m < n$	$\frac{\frac{a}{b}}{x^{n-m}}$	Horizontal asymptote $y = 0$
$m = n$	$\frac{a}{b}$	Horizontal asymptote $y = \frac{a}{b}$
$m = n + 1$	$\left(\frac{a}{b}\right)x$	Slant slope $\frac{a}{b}$; divide for intercept
$m > n + 1$	$\left(\frac{a}{b}\right)x^{m-n}$	Unbounded ends; no line asymptote

EXAMPLE 1.7.5 | An average cost model

A project costs $C(n) = 1200 + 8n$ dollars for n books. Interpret $A(n) = \frac{C(n)}{n}$ for positive whole-number n.

Solution

$A(n) = 8 + \frac{1200}{n}$ dollars per book. Its mathematical extension has horizontal asymptote $y = 8$.

As more books share the fixed cost, the average cost approaches \$8 per book from above. At $n = 100$ the average is \$20; at $n = 300$ it is \$12.

The model assumes a fixed \$1200 setup cost and constant \$8 variable cost per book. Capacity constraints could limit the useful domain.

CHECK 1.7.C4

For $A(n) = 8 + \frac{1200}{n}$ with $n > 0$, can $A(n) = 8$ occur at a finite n? Explain.

Calculator check: compare a moderate window with a much wider one. A table at both positive and negative large inputs checks the full mathematical domain; a real-world context may use only one of those directions.

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1.8 Rational Functions and Zeros

FOCUS Find numerator zeros that belong to the original domain. • Solve rational inequalities using all numerator and denominator zeros as boundaries.

A zero must be a legal input

A quotient is zero exactly when its numerator is zero and its denominator is nonzero. Start by recording exclusions from the original denominator. Then solve the numerator equation and discard any candidate outside that original domain.

EXAMPLE 1.8.1 | A canceled factor does not create a zero

Find the zeros of $r(x) = \frac{x^2-2x-3}{x^2-x-2}$.

Solution

Factor: $r(x) = \frac{(x-3)(x+1)}{(x-2)(x+1)}$. The original domain excludes -1 and 2.

The numerator is zero at -1 and 3. The input -1 is excluded, so the only zero is 3.

The reduced expression is $\frac{x-3}{x-2}$, but the exclusion $x \neq -1$ must still be retained.

The y-intercept, when it exists, is $(0, r(0))$. It is found by substituting 0, not by setting the numerator equal to 0. An x-intercept has output 0; a y-intercept has input 0.

EXAMPLE 1.8.2 | Identify both intercepts

Find the intercepts of the function in Example 1.8.1.

Solution

The x-intercept is $(3, 0)$. Since 0 is in the original domain, $r(0) = \frac{-3}{-2} = \frac{3}{2}$.

Thus its y-intercept is $\left(0, \frac{3}{2}\right)$. The excluded input -1 is neither an x-intercept nor a valid function value.

CHECK 1.8.C1

Find the zeros of $\frac{(x-4)(x+2)}{(x-4)(x-1)}$.

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Build a rational sign chart

To solve $r(x) > 0, r(x) < 0, r(x) \geq 0$, or $r(x) \leq 0$, mark every real numerator zero and denominator zero. These split the number line into intervals on which the sign is constant. A canceled factor still marks a domain exclusion.

Interval for r in Example 1.8.1	Test input	Sign of $\frac{x-3}{x-2}$
$(-\infty, -1)$	-2	+
$(-1, 2)$	0	+
$(2, 3)$	$\frac{5}{2}$	-
$(3, \infty)$	4	+

EXAMPLE 1.8.3 | Include zeros; exclude undefined inputs

Solve $r(x) \geq 0$ for $r(x) = \frac{x^2-2x-3}{x^2-x-2}$.

Solution

The positive intervals are $(-\infty, -1), (-1, 2)$, and $(3, \infty)$. The only legal zero is 3, so include 3.

The solution is $(-\infty, -1) \cup (-1, 2) \cup [3, \infty)$. Neither -1 nor 2 can be included.

For the strict inequality $r(x) < 0$, the solution would be $(2, 3)$.

Why not multiply by the denominator?

Its sign can change with x. Multiplying an inequality by a negative quantity reverses the inequality. A sign chart handles those cases systematically.

CHECK 1.8.C2

Solve $\frac{x-1}{x+2} \leq 0$.

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Use a graph as an independent representation

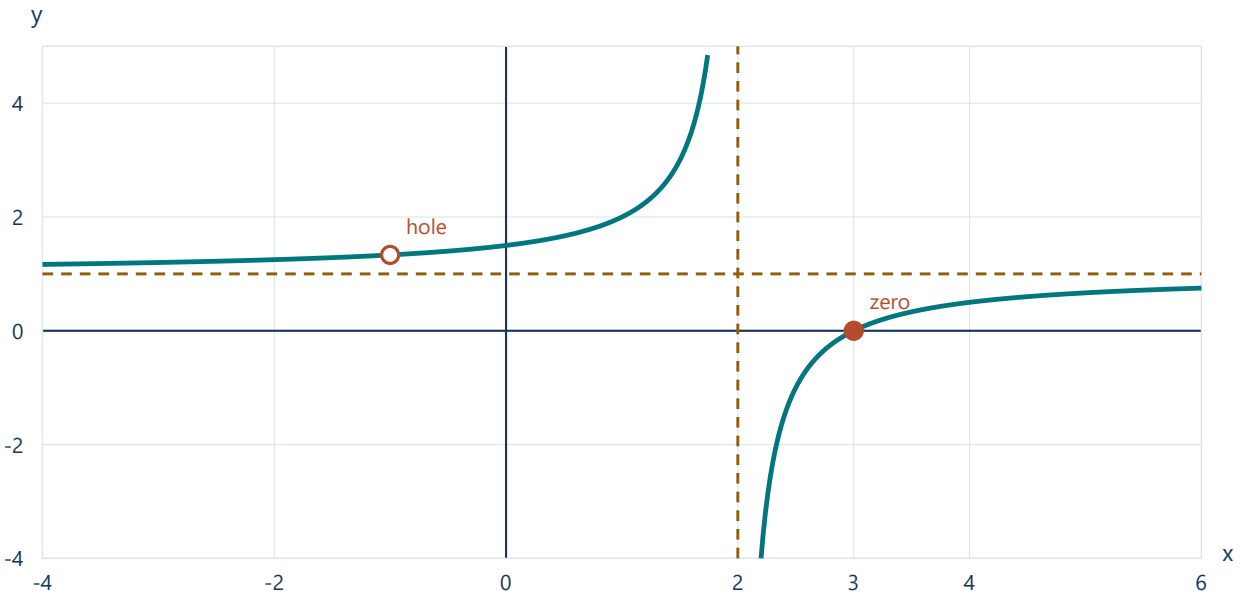


Figure 1.8.1. The graph of the original rational function has a hole at $\left(-1, \frac{4}{3}\right)$, a vertical asymptote $x = 2$, and its only zero at 3.

A graph below the horizontal axis corresponds to negative outputs. A graph above it corresponds to positive outputs. The hole is an unfilled point even though the nearby curve is above the axis.

EXAMPLE 1.8.4 | Explain an incorrect solution

A student simplifies $r(x)$, obtains a positive value $\frac{4}{3}$ at $x = -1$, and includes -1 in the inequality solution. What failed?

Solution

The student evaluated the extension $\frac{x-3}{x-2}$, not the original function. The original expression is undefined at -1.

Equality of formulas after cancellation is valid only on the original domain. A positive extension value does not make an excluded input legal.

A calculator display may not show a one-point hole. Preserve the original algebraic domain even when the graphing software draws what looks like an uninterrupted curve.

CHECK 1.8.C3

Why is $x = 2$ excluded from the solution to $\frac{(x-2)^2}{x-2} \geq 0$, even though the reduced expression has value 0 there?

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Repeated factors and a reliable procedure

EXAMPLE 1.8.5 | A repeated numerator zero

Solve $\frac{(x-1)^2}{x+3} > 0$.

Solution

The numerator is positive except at 1, where it is 0. The denominator is positive for $x \succ 3$ and negative for $x < -3$.

For strict positivity, require $x \succ 3$ and $x \neq 1$. The solution is $(-3, 1) \cup (1, \infty)$.

If the inequality were ≥ 0, then 1 would be included, giving $(-3, \infty)$. The excluded denominator input -3 is never included.

Four checks before finishing

Record original exclusions. List all candidate zeros. Test signs on every interval. Decide endpoint inclusion separately from interval signs.

When the task asks for a simplified expression rather than the full domain, explicitly state restrictions lost through cancellation. For a request to identify every excluded input or give the full domain, list all original denominator zeros. This distinction prevents a correct simplification from being mistaken for a change in the function.

CHECK 1.8.C4

Solve $\frac{(x+1)^2}{x-2} < 0$.

Summary: zeros describe legal inputs with output 0. Undefined inputs remain excluded for both equations and inequalities, even after a common factor cancels.

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1.9 Rational Functions and Vertical Asymptotes

FOCUS Identify vertical asymptotes after comparing multiplicities. • Determine each one-sided unbounded limit using signs.

A denominator zero is a candidate

A vertical asymptote occurs at $x = a$ when outputs become unbounded on at least one side as x approaches a. For a rational function, factor numerator and denominator and compare the multiplicity of the common factor $x - a$.

Let m be its numerator multiplicity and n its denominator multiplicity, with $m = 0$ if it is absent from the numerator. If $n > m$, a denominator factor remains after reduction and $x = a$ is a vertical asymptote. If $n \leq m$ and $n > 0$, the input is a hole instead; Topic 1.10 explains the finite limit.

EXAMPLE 1.9.1 | A common factor can still leave an asymptote

Classify excluded inputs for $r(x) = \frac{(x-1)^2}{(x-1)^3(x+2)}$.

Solution

The original domain excludes 1 and -2. Reduction gives $r(x) = \frac{1}{(x-1)(x+2)}$ on that domain.

At 1 the denominator multiplicity 3 exceeds the numerator multiplicity 2. One denominator factor remains, so $x = 1$ is a vertical asymptote.

At -2 the numerator multiplicity is 0 and denominator multiplicity is 1. It is also a vertical asymptote.

CHECK 1.9.C1

At $x = 4a$ numerator has multiplicity 2 and a denominator has multiplicity 5. Is there a vertical asymptote?

Do not stop at 'both are zero'

A $\frac{0}{0}$ substitution is not a classification. Compare how many copies of the factor occur on each side of the fraction.

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Determine the side before the sign

The notation $x \to a^-$ means approach a using inputs less than a. The notation $x \to a^+$ means approach using inputs greater than a. The superscripts refer to the side of the input, not the sign of the output.

EXAMPLE 1.9.2 | Two sides of the same vertical line

Find the one-sided limits of $r(x) = \frac{1}{(x-1)(x+2)}$ at $x = 1$.

Solution

Near 1, $x + 2$ stays positive. To the left, $x - 1$ is negative, so the denominator is a small negative number and $r(x) \to -\infty$.

To the right, both factors are positive and the denominator is a small positive number, so $r(x) \to \infty$.

Write $\lim_{x \to 1^-} r(x) = -\infty$ and $\lim_{x \to 1^+} r(x) = \infty$.

Input x	0.99	0.999	1.001	1.01
$r(x)$, approximately	-33.445	-333.444	333.222	33.223

The table supports the sign analysis. A few large values alone are not a proof of unbounded behavior; the factored expression explains why the denominator shrinks while the numerator remains nonzero.

CHECK 1.9.C2

Find the left and right limits of $\frac{2}{x+3}$ at $x = -3$.

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Even and odd poles

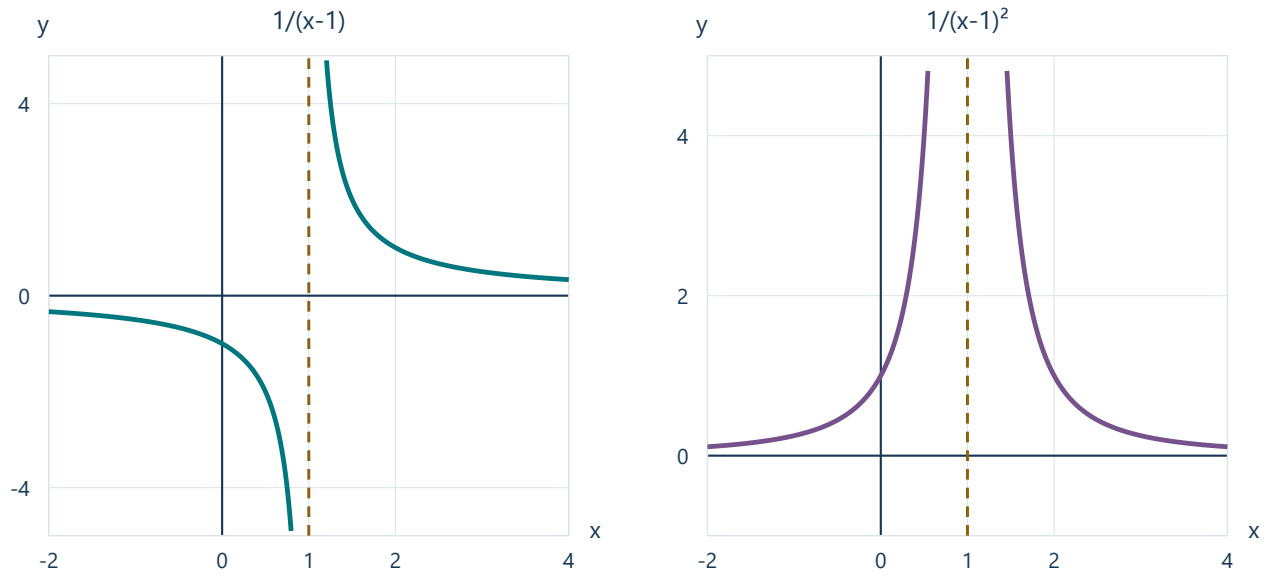


Figure 1.9.1. An odd remaining denominator power gives opposite signs across the asymptote; an even remaining power gives matching signs.

EXAMPLE 1.9.3 | Same direction on both sides

Find the one-sided limits of $r(x) = -\frac{2}{(x+1)^2}$ at $x = -1$.

Solution

The squared denominator is positive on both sides and approaches 0. The numerator remains -2.

Both one-sided limits are $-\infty$. A vertical asymptote does not require opposite-side behavior.

Near an asymptote, write the function as $\frac{g(x)}{(x-a)^k}$, where g stays nonzero and keeps its sign. An odd k changes the denominator sign across a. An even k does not. The sign of g decides whether the outputs are positive or negative.

CHECK 1.9.C3

For $\frac{5}{(x-2)^4}$, what are the one-sided limits at 2?

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All asymptotes, all sides

EXAMPLE 1.9.4 | The other excluded input

For $r(x) = \frac{1}{(x-1)(x+2)}$, find the one-sided limits at $x = -2$.

Solution

Near -2, $x - 1$ stays negative. Left of -2, $x + 2$ is negative, so the denominator is positive and $r(x) \to \infty$.

Right of -2, $x + 2$ is positive, so the denominator is negative and $r(x) \to -\infty$.

Together with Example 1.9.2, this specifies all four approaches to the two vertical asymptotes.

A graphing program may join points across a discontinuity or draw steep branches poorly. Check algebra, use separate input intervals, and inspect a table near the suspected asymptote. For a rational function on its natural domain, the asymptote input itself is undefined.

EXAMPLE 1.9.5 | Context limits what an asymptote means

A model $T(v) = \frac{120}{v}$ gives travel time in hours for a 120 km trip at constant speed $v$km/h. Interpret $v \to 0^+$.

Solution

The context requires $v > 0$. As positive speed approaches 0, the model's travel time increases without bound.

The relevant statement is $\lim_{v \to 0^+} T(v) = \infty$. Negative speeds in the algebraic extension are not part of this constant-speed travel-time context.

CHECK 1.9.C4

Can canceling one common factor guarantee a hole? Give a counterexample.

Summary

Compare multiplicities first. Then determine the sign on each side separately. Keep the original excluded input out of the domain.

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1.10 Rational Functions and Holes

FOCUS Find holes and their coordinates using the original domain and a reduced expression. • Distinguish function value, limit, and a filled-in extension.

An excluded input with a finite nearby value

A hole is a missing graph point at an excluded input where nearby outputs approach a finite value. At a real denominator zero a, compare multiplicities. If the numerator has at least as many factors $x - a$ as the denominator, all denominator copies cancel. The reduced expression has a finite value at a, but the original function still does not.

EXAMPLE 1.10.1 | Find both coordinates

Locate the hole in $r(x) = \frac{x^2-x-2}{x-2}$.

Solution

Factor the numerator as $(x - 2)(x + 1)$. For $x \neq 2, r(x) = x + 1$.

The reduced expression approaches 3 as x approaches 2. The hole is at $(2, 3)$, and $\lim_{x\to 2} r(x) = 3$.

The original function value $r(2)$ is undefined. A limit can exist at an input where the function has no value.

$$r(x) = \frac{(x-2)(x+1)}{x-2} = x + 1, x \neq 2$$

CHECK 1.10.C1

Locate the hole in $\frac{x^2-9}{x-3}$.

Two coordinates are required

The excluded input gives the x-coordinate. Evaluate the fully reduced expression there to obtain the y-coordinate.

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The graph and the limit

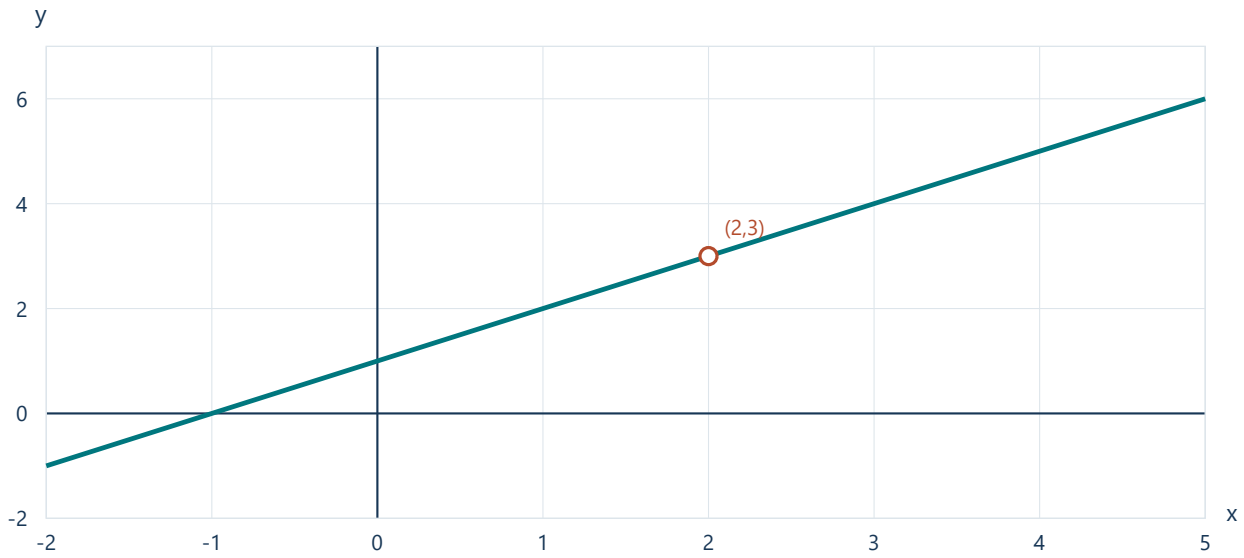


Figure 1.10.1. The graph follows $y = x + 1$ but omits $(2, 3)$. An open point marks the hole.

x	1.9	1.99	2	2.01	2.1
Original $r(x)$	2.9	2.99	Undefined	3.01	3.1

Both sides approach the same output. Consequently $\lim_{x \to 2^-} r(x) = 3$, $\lim_{x \to 2^+} r(x) = 3$, and $\lim_{x \to 2} r(x) = 3$.

The limit describes nearby behavior; it does not insert a value into the function's domain.

EXAMPLE 1.10.2 | Fill in the missing input

Define $g(x) = x + 1$ for every real x. Compare g and the function in Example 1.10.1.

Solution

The two functions have the same output at every input in r's domain. However, they are not equal functions because their domains differ.

The extension g has $g(2) = 3$ and fills the hole. It is continuous there; the original r remains undefined.

CHECK 1.10.C2

A function has a hole at $(5, -2)$. What is its limit as x approaches 5? Is $f(5) = -2$?

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A hole can lie on the horizontal axis

EXAMPLE 1.10.3 | Excess numerator multiplicity

Analyze $r(x) = \frac{(x-1)^3}{(x-1)(x+2)}$ at its excluded inputs.

Solution

The original domain excludes 1 and -2. Reduction gives $\frac{(x-1)^2}{x+2}$.

At 1 all denominator copies cancel and the reduced value is 0. There is a hole at $(1, 0)$, not a zero of the original function.

At $-2$$a$ denominator factor remains and the numerator is 9, so $x = -2$ is a vertical asymptote.

A point on the x-axis is an intercept only if the function includes it. A missing point with output coordinate 0 is still missing. The graph of this particular function has no real zero because its only candidate, 1, is excluded.

CHECK 1.10.C3

For $\frac{(x+4)^2}{x+4}$, is $x = -4$$a$ zero, a hole, or a vertical asymptote? Give the point when appropriate.

Multiplicity checklist

Numerator multiplicity smaller: vertical asymptote. Equal: hole with a nonzero reduced value, provided the other factors are nonzero. Greater: hole with reduced value 0.

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Several excluded inputs

EXAMPLE 1.10.4 | Separate the roles of two factors

Find the holes and vertical asymptotes of $r(x) = \frac{(x-2)(x+1)}{(x-2)(x-3)}$.

Solution

The original domain excludes 2 and 3. Reduction gives $\frac{x+1}{x-3}$.

At 2, the extension value is $\frac{3}{-1} = -3$, so the hole is $(2, -3)$. At 3, a denominator zero remains and the numerator is 4, so $x = 3$ is a vertical asymptote.

There is a valid zero at $x = -1$ because -1 is in the original domain.

Use a factored expression for classification, the reduced expression for a finite hole height, and the original expression for the domain. These are three jobs served by two representations of the same original rule.

CHECK 1.10.C4

For $r(x) = \frac{x^2-4}{x^2-x-2}$, find all exclusions, the hole, and any vertical asymptote.

Calculator warning

A one-point hole may be invisible at any ordinary zoom. Draw the open point deliberately when communicating the graph. A graph that looks like a complete line can still represent a function with a missing input.

Summary: cancellation changes the expression used for legal inputs, not the original domain. The hole height is a limit; it is not an original function value.

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1.11 Equivalent Representations of Polynomial and Rational Expressions

FOCUS Choose a form that exposes the needed feature. • Use polynomial long division and identify slant asymptotes. • Expand repeated binomial products using Pascal's triangle or the binomial theorem.

Choose a form for a purpose

Equivalent expressions can make different features easy to see. A factored form exposes zeros and common factors. Standard form exposes degree, leading coefficient, and the constant term. Vertex form of a quadratic exposes an extreme value and axis of symmetry.

EXAMPLE 1.11.1 | One quadratic, three views

Rewrite $f(x) = x^2 - 6x + 5$ in factored and vertex form. Use each form to describe a feature.

Solution

Factoring gives $f(x) = (x - 1)(x - 5)$, so its zeros are 1 and 5.

Completing the square gives $f(x) = (x - 3)^2 - 4$, so its vertex is $(3, -4)$, axis is $x = 3$, and range is $[-4, \infty)$.

Standard form shows a positive leading coefficient and degree 2, so both ends rise. It also gives the y-intercept $(0, 5)$.

$$x^2 - 6x + 5 = (x - 1)(x - 5) = (x - 3)^2 - 4$$

Completing the square

Half the coefficient of x is -3. Add and subtract its square, 9:
$x^2 - 6x + 9 - 9 + 5 = (x - 3)^2 - 4$. The added and subtracted amounts preserve equality.

CHECK 1.11.C1

Write $x^2 + 4x - 5$ in factored and vertex forms.

Equivalent algebraic expressions define equal functions only when the domains also agree. Keep domain restrictions attached to rational simplifications.

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Interpret the representation in context

EXAMPLE 1.11.2 | Revenue and a maximum

A model for ticket revenue is $R(p) = -2p^2 + 80p$ dollars when the ticket price is p dollars, $0 \le p \le 40$. Find the maximum modeled revenue and explain both zeros.

Solution

Factored form $R(p) = 2p(40 - p)$ shows zeros at $p = 0$ and $p = 40$. A free ticket gives no ticket revenue; the model predicts zero demand at the upper price.

Vertex form $R(p) = -2(p - 20)^2 + 800$ shows maximum revenue \$800 at a price of \$20.

The implied demand is $80 - 2p$ tickets. The domain keeps this demand nonnegative. The model treats demand continuously; an actual ticket count is a whole number.

A good interpretation names the input, output, units, and restrictions. A vertex outside the contextual domain would not automatically give the relevant maximum; compare values on the permitted interval.

EXAMPLE 1.11.3 | Add rational expressions

Simplify $\frac{1}{x-1} + \frac{2}{x+1}$, retaining its domain.

Solution

Use the common denominator $(x - 1)(x + 1)$. The numerator becomes $(x + 1) + 2(x - 1) = 3x - 1$.

The result is $\frac{3x-1}{x^2-1}$, with $x \neq -1, 1$. Adding denominators directly would not preserve equality.

CHECK 1.11.C2

Simplify $\frac{x^2-4}{x^2-x-2}$ and list every original exclusion.

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Polynomial long division

Polynomial division produces a quotient q and remainder r satisfying $f = gq + r$, where the remainder has smaller degree than the divisor. Arrange both polynomials by descending powers and insert zero terms for missing powers. Divide the leading terms, multiply, subtract, and repeat.

EXAMPLE 1.11.4 | Keep the missing x term

Divide $2x^3 - 3x^2 + 0x + 4$ by $x - 2$.

Solution

First: $\frac{2x^3}{x} = 2x^2$. Subtract $(x-2)\left(2x^2\right) = 2x^3 - 4x^2$, leaving $x^2 + 0x + 4$.

Next: $\frac{x^2}{x} = x$. Subtract $(x-2)x = x^2 - 2x$, leaving $2x + 4$.

Next: $\frac{2x}{x} = 2$. Subtract $(x-2)2 = 2x - 4$, leaving remainder 8.

The quotient is $2x^2 + x + 2$ and the remainder is 8. Thus the rational expression is $2x^2 + x + 2 + \frac{8}{x-2}$, for $x \neq 2$.

Stage	Quotient term	Expression subtracted	Remainder so far
1	$2x^2$	$2x^3 - 4x^2$	$x^2 + 0x + 4$
2	x	$x^2 - 2x$	$2x + 4$
3	2	$2x - 4$	8

Independent check

Multiply $(x-2)\left(2x^2 + x + 2\right) + 8$. It expands to $2x^3 - 3x^2 + 4$, the original dividend. Check both the coefficients and the remainder-degree condition.

CHECK 1.11.C3

Divide $x^2 + 3x + 5$ by $x + 1$. Give quotient and remainder.

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Division reveals a slant asymptote

When the numerator degree is exactly one more than the denominator degree, division gives a linear quotient plus a remainder fraction that approaches 0 at either end. The linear quotient is the slant asymptote.

EXAMPLE 1.11.5 | Recover the full line

Find the slant asymptote of $r(x) = \frac{2x^2+3x+5}{x+1}$.

Solution

Divide to obtain $r(x) = 2x + 1 + \frac{4}{x+1}, x \neq -1$.

As $x \to \infty$ or $x \to -\infty$, $\frac{4}{x+1} \to 0$. The vertical difference between the rational graph and $y = 2x + 1$ approaches 0.

The slant asymptote is $y = 2x + 1$, not $y = 2x$. The leading-term ratio alone gives only its slope.

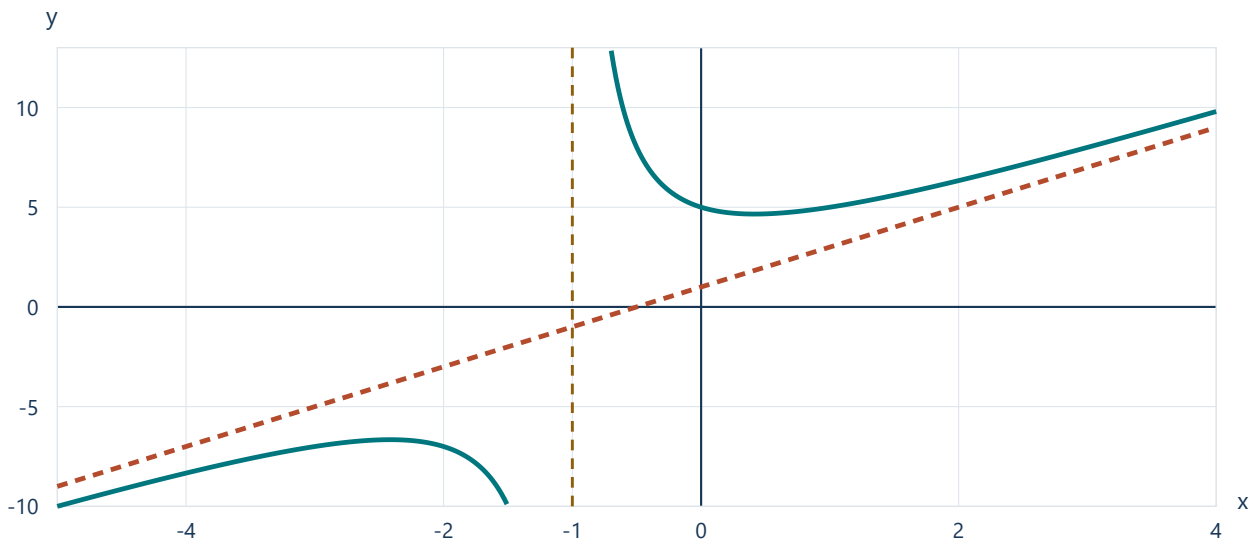


Figure 1.11.1. The rational curve approaches the dashed line $y = 2x + 1$. The vertical asymptote is $x = -1$.

CHECK 1.11.C4

Find the slant asymptote of $\frac{x^2+1}{x-1}$.

Division also helps interpret rational models by separating a dominant trend from a smaller correction. It does not erase any original input exclusions.

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Pascal's triangle and repeated products

Repeated binomial products have a predictable coefficient pattern. Each row of Pascal's triangle begins and ends with 1; each interior entry is the sum of the two entries above it. Row n supplies the coefficients of $(a+b)^n$. Number the top row as row 0.

n	Coefficients in row n
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

EXAMPLE 1.11.6 | Include powers of both terms

Expand $(2x-3)^4$.

Solution

Use $a = 2x$ and $b = -3$. Row 4 supplies $1, 4, 6, 4, 1$.

The five terms are $(2x)^4, 4(2x)^3(-3), 6(2x)^2(-3)^2, 4(2x)(-3)^3$, and $(-3)^4$.

After simplification: $16x^4 - 96x^3 + 216x^2 - 216x + 81$. Negative b causes alternating signs.

CHECK 1.11.C5

Expand $(x+2)^3$.

Common error

Pascal's entries are coefficients before substituting a and b. They do not replace the powers of 2 or 3 in $(2x-3)^4$.

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The general binomial theorem

For a nonnegative integer n, the coefficient sequence is given by the binomial coefficients $C(n,k) = \frac{n!}{k!(n-k)!}$, for $k = 0,\ldots,n$. Each term is $C(n,k)a^{n-k}b^k$. The exponent on a decreases while the exponent on b increases, and their sum stays n.

EXAMPLE 1.11.7 | Find one coefficient without the whole expansion

Find the coefficient of x^3 in $(x+2)^5$.

Solution

The x^3 term uses $a = x$ and exponent $5 - k = 3$, so $k = 2$.

Its coefficient is $C(5,2)2^2 = 10 \cdot 4 = 40$. The term is $40x^3$.

Pascal row 5 gives the same coefficient entry 10 before multiplying by 2^2.

If factorial notation is unfamiliar, use the appropriate row of Pascal's triangle for small powers. A coefficient question can be answered by identifying its one term rather than expanding every term.

CHECK 1.11.C6

Find the coefficient of x^2 in $(x-1)^4$.

A representation checklist

Factored: zeros and cancellations. Standard: degree, leading sign, constant term. Vertex: a quadratic extreme. Quotient plus remainder: asymptotic line. Binomial expansion: coefficients of repeated products.

Keep for Calculus: useful equivalent forms can make a difficult expression easier to interpret before any new technique is applied. Algebraic equivalence remains conditional on retaining the original domain.

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1.12 Transformations of Functions

FOCUS Construct additive and multiplicative transformations. • Map points, domains, and ranges, including combined horizontal changes.

Vertical and horizontal translations

A vertical translation changes outputs: $g(x) = f(x) + k$ sends (u, v) to $(u, v + k)$. A horizontal translation changes the input needed to get an old output: $g(x) = f(x - h)$ sends (u, v) to $(u + h, v)$.

To understand a horizontal change, solve the new inside expression = the old input. If $f(2) = 7$ and $g(x) = f(x - 3)$, the output 7 occurs when $x - 3 = 2$, so $x = 5$. The point moves right by 3.

EXAMPLE 1.12.1 | Read the sign inside

Describe $g(x) = f(x + 4) - 2$.

Solution

Solve $x + 4 = u$ to get $x = u - 4$: every point moves 4 units left. Subtracting 2 outside moves it 2 units down.

Thus (u, v) maps to $(u - 4, v - 2)$. In particular, $(3, 5)$ on f maps to $(-1, 3)$ on g.

CHECK 1.12.C1

If $f(-2) = 6$ and $g(x) = f(x - 5) + 1$, give the corresponding point on g.

A reliable alternative to a slogan

Rather than memorizing 'inside is opposite,' solve for the new input. That method also works when a horizontal scale and shift appear together.

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Dilations and reflections

In $g(x) = af(x)$, outputs are multiplied by a. The vertical scale factor is $|a|$, and $a < 0$ also reflects the graph across the horizontal axis. In $g(x) = f(bx)$, solve $bx = u$ to get $x = \frac{u}{b}$. The horizontal scale factor is $\frac{1}{|b|}$, with a reflection across the vertical axis when $b < 0$.

Transformation	Point mapping
$f(x) + k$	$(u, v) \to (u, v + k)$
$f(x - h)$	$(u, v) \to (u + h, v)$
$af(x)$	$(u, v) \to (u, av)$
$f(bx)$	$(u, v) \to \left(\frac{u}{b}, v\right)$

EXAMPLE 1.12.2 | Different effects of the same number

Compare $g(x) = 2f(x)$ and $h(x) = f(2x)$ using a point $(4, 3)$ on f.

Solution

For g, the output doubles: the corresponding point is $(4, 6)$.

For h, the input halves: the corresponding point is $(2, 3)$. It takes half the horizontal distance to reach the same old output.

These transformations are generally different. They can produce identical formulas for special parent functions, so explain the operation rather than relying only on appearance.

CHECK 1.12.C2

Map $(6, -2)$ under $g(x) = -3f(-2x)$.

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Combined transformations and point mapping

A useful combined form is $g(x) = af(b(x-h)) + k$ with $a, b \neq 0$. Solve $b(x-h) = u$ and transform the output $av + k$. The full point mapping is

$$(u, v) \mapsto \left(h + \frac{u}{b},\ av + k\right)$$

EXAMPLE 1.12.3 | Factor the inside first

For $g(x) = -2f(3x-3) + 4$, map an original point $(2, 5)$.

Solution

Factor the input expression: $3x - 3 = 3(x-1)$. Thus $h = 1$ and $b = 3$. Map the input and output separately.

$$x_{\text{new}} = 1 + \frac{2}{3} = \frac{5}{3}, \qquad y_{\text{new}} = -2(5) + 4 = -6.$$

Therefore the corresponding point is $\left(\frac{5}{3}, -6\right)$.

The graph is horizontally compressed by $\frac{1}{3}$, shifted right 1, vertically stretched by 2 and reflected across the horizontal axis, then shifted up 4.

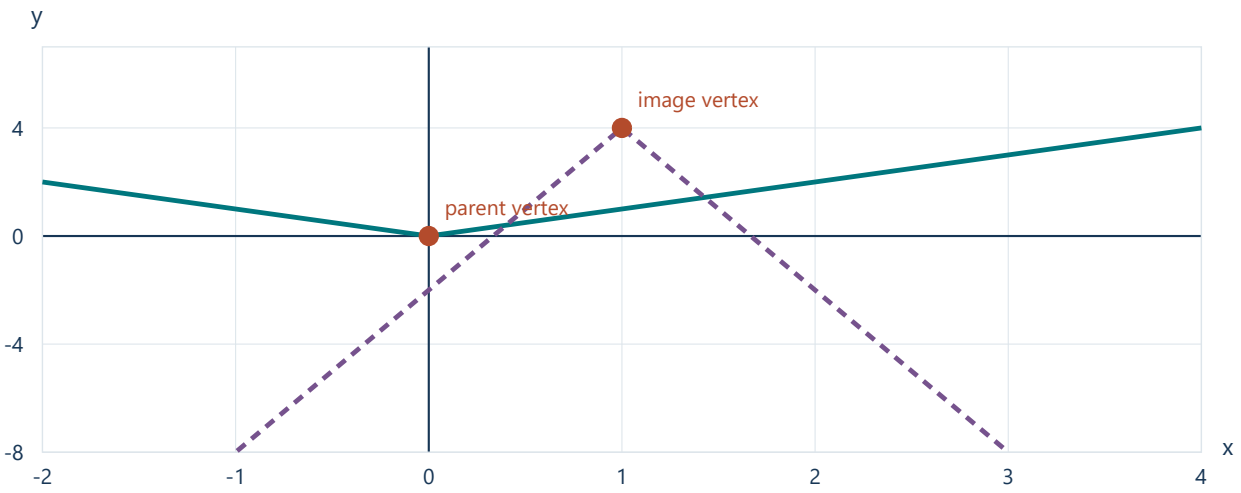


Figure 1.12.1. The solid parent $|x|$ and dashed image $-2|3(x-1)| + 4$ show both input and output changes.

CHECK 1.12.C3

In $f(2x-6)$, is the horizontal shift right 6 or right 3? Explain.

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Transform the domain and the range

A transformed formula is defined only when its inside expression belongs to the original domain. For $g(x) = af(b(x - h)) + k$, transform every allowed original input u by $x = h + \frac{u}{b}$. Transform every allowed original output v by $y = av + k$. Negative scales reverse interval order.

EXAMPLE 1.12.4 | A domain and range without a formula

Suppose f has domain $[-2, 4]$ and range $[-1, 3]$. Find the domain and range of $g(x) = -2f(-3(x - 1)) + 4$.

Solution

The input map is $x = 1 - \frac{u}{3}$. The original endpoints -2 and 4 map to $\frac{5}{3}$ and $-\frac{1}{3}$, respectively. Put them in increasing order: domain $\left[-\frac{1}{3}, \frac{5}{3}\right]$.

The output map is $y = -2v + 4$. Outputs -1 and 3 map to 6 and -2, respectively, so the range is $[-2, 6]$.

Included endpoints stay included. No equation for f is needed.

CHECK 1.12.C4

If f has domain $[0, 8]$ and range $[1, 5]$, find the domain and range of $g(x) = 3f(2x - 6) - 1$.

For a rational parent, transform excluded inputs as well. If f is undefined at $u = 0$, then $f(2(x - 3))$ is undefined at $x = 3$. Domain restrictions move with the horizontal transformation.

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Tables and graphs are enough

EXAMPLE 1.12.5 | Transform a table

A function is defined only by the pairs $(-2, 4), (0, 1), (3, 2)$. Find all pairs for $g(x) = 2f(x + 1) - 3$.

Solution

The input map is $x = u - 1$ and output map $y = 2v - 3$.

The transformed pairs are $(-3, 5), (-1, -1)$, and $(2, 1)$. The new domain is $\{-3, -1, 2\}$ and range is $\{-1, 1, 5\}$.

Do not connect these three points with a curve unless an additional continuous-domain rule is given.

Old input u	Old output v	New input $u - 1$	New output $2v - 3$
-2	4	-3	5
0	1	-1	-1
3	2	2	1

In transformation language, the original set of points is the preimage and the transformed set is the image. Topic 1.1 also used image for the output of one input and preimage for all inputs producing a selected output. Read which mapping the question refers to.

CHECK 1.12.C5

A graph includes $(2, -1)$. Give its image under $g(x) = f\left(\frac{x}{2}\right) + 3$.

Check a transformation

Choose a point whose old value is known. Substitute the new input into the new expression and verify the required output.

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Construct a transformed model

EXAMPLE 1.12.6 | Build from a vertex and one point

A quadratic has vertex $(3, 5)$ and passes through $(1, -3)$. Construct its equation.

Solution

Start with the parent $f(x) = x^2$ and write $g(x) = a(x - 3)^2 + 5$.

Substitute $(1, -3)$: $-3 = 4a + 5$, so $a = -2$. The model is $g(x) = -2(x - 3)^2 + 5$.

Its range is $(-\infty, 5]$ on the full real domain, and its graph is reflected vertically and stretched by a factor of 2.

EXAMPLE 1.12.7 | Change the time scale

A process has output $f(t)$ at elapsed time t. A second process follows the same output pattern twice as quickly. Give a model if both begin at $t = 0$.

Solution

Use $g(t) = f(2t)$. The old output at time u occurs at the new time $\frac{u}{2}$.

Using $2f(t)$ would double output levels instead of accelerating the timing. The appropriate model depends on what the context says changes.

CHECK 1.12.C6

Construct a quadratic with vertex $(-1, 2)$ through $(1, 10)$.

Summary: factor the input expression, map a known point, transform domain and range, and interpret the result in context. A list of operations is useful only if it reproduces the intended point mapping.

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1.13 Function Model Selection and Assumption Articulation

FOCUS Select a plausible function family using context, rates, and differences. • State assumptions about consistency, change, domain, and range. • Distinguish interpolation from an explanatory model.

Choose a family before fitting numbers

A model is a function used to represent selected features of a situation. Choosing a model requires a mathematical pattern and a reason to expect that pattern in the context. A good fit to a few points is useful evidence, but it is not proof that the same pattern continues indefinitely.

Evidence or mechanism	Plausible starting family
Roughly constant rate	Linear
Roughly linearly changing rate; one symmetric peak or trough	Quadratic
Three changing dimensions; volume	Often cubic
Several real zeros or turning points	A suitable polynomial
Different rules in different intervals	Piecewise-defined
Constant product of input and output	Inverse variation, $\frac{k}{x}$

The words often and roughly matter. An area need not be quadratic if one dimension is fixed; a volume need not be cubic if only one dimension varies. Write the relationships between dimensions instead of choosing a degree from a noun alone.

EXAMPLE 1.13.1 | Use the mechanism

Compare a rectangle of width 5 and length x with a square of side x.

Solution

The rectangle has $A(x) = 5x$, a linear model. The square has $A(x) = x^2$, a quadratic model.

Both contexts concern area. The difference is which dimensions change with x and how they change.

CHECK 1.13.C1

A cube has side length $s > 0$. Which polynomial family describes volume? What assumption connects the dimensions?

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Patterns in rates and differences

EXAMPLE 1.13.2 | Recognize a quadratic with nonunit spacing

Inputs are $0, 2, 4, 6, 8$ and outputs are $1, 9, 25, 49, 81$. Select a least-degree polynomial model.

Solution

The equal input step is 2. First output differences are $8, 16, 24, 32$. Second differences are $8, 8, 8$, constant and nonzero.

A quadratic is a plausible least-degree polynomial model. The rule $y = (x + 1)^2$ reproduces all five pairs.

The associated average rates are $4, 8, 12, 16$, which change linearly as the inputs progress.

x	0	1	2	3	4	5
A quartic pattern: x^4	0	1	16	81	256	625
First differences		1	15	65	175	369
Second differences			14	50	110	194
Third differences				36	60	84
Fourth differences					24	24

Real measurements usually contain noise, so differences need not be exactly constant. Higher-order differences can magnify small measurement errors. Use the context and graph as well as the table; avoid fitting a high degree merely to chase noisy differences.

CHECK 1.13.C2

Equal-step data have first differences $4, 9, 14, 19$. What least-degree polynomial family is suggested?

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Enough points to interpolate does not prove a law

There is a unique polynomial of degree at most n through $n+1$ points with distinct input values. The words at most matter: the fitted degree may be lower. Distinct inputs also matter: two different outputs at the same input cannot both belong to a function.

EXAMPLE 1.13.3 | Three points do not force degree two

A data set is $(0,1),(1,3),(2,5)$. What does the interpolation statement guarantee?

Solution

Three distinct inputs guarantee a polynomial of degree at most 2. Here the linear polynomial $2x+1$ already fits all three points, so the least degree is 1.

The cubic $2x+1+x(x-1)(x-2)$ also fits those points because its added product is zero at each sampled input.

A degree restriction selects the unique interpolating polynomial within that restricted family; the data do not select one function from all possible functions.

An exact interpolating model is appropriate when constraints are exact and the selected structure is justified. For noisy observations, a regression model usually should not be forced through each point.

CHECK 1.13.C3

How many distinct input-output pairs determine a unique polynomial of degree at most 4? Does that guarantee the degree is exactly 4?

Useful skepticism

Ask what changes if one measurement is slightly wrong. A model that changes wildly under a tiny perturbation may be a poor choice for prediction.

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Geometry and multiple features

EXAMPLE 1.13.4 | Build the family from dimensions

A 12 cm by 20 cm sheet has squares of side x cut from its corners and is folded into an open box. Choose a volume model and domain.

Solution

The height is x, width $12 - 2x$, and length $20 - 2x$. Thus
$V(x) = x(12 - 2x)(20 - 2x) = 4x^3 - 64x^2 + 240x$.

All three dimensions must be positive, so $0 < x < 6$. This is a cubic model because three linear factors vary together.

The algebraic formula exists outside that interval, but negative side lengths or a flattened box do not represent the stated physical object.

Distinct real zeros, repeated contacts with an axis, or several peaks and troughs can suggest a polynomial degree and factor structure. They usually give a minimum possible degree, not an exact one, unless further assumptions are supplied.

EXAMPLE 1.13.5 | Constrain a least-degree model

A polynomial model has zeros at -2 and 3, touches at 1, and has positive outputs for large positive inputs. Propose a least-degree factored form.

Solution

Use odd multiplicity at the crossing zeros and even multiplicity at the touch. The least-degree form is $k(x + 2)(x - 1)^2(x - 3)$, with $k > 0$.

A value at another legal input is needed to determine k.

CHECK 1.13.C4

For the box model, can $x = 7\text{cm}$ be used as a physical input? Why?

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When the rule changes

A piecewise-defined model uses different function rules on nonoverlapping parts of its domain. Each permitted input must receive exactly one output. The rules may join continuously, or the context may justify a jump.

EXAMPLE 1.13.6 | A change in pricing rule

A taxi charges \$5 to begin a trip, then \$2 per kilometer for the first 10 km and \$1.50 per kilometer beyond 10 km. Choose a piecewise model for $d \geq 0$.

Solution

For $0 \leq d \leq 10$, $C(d) = 5 + 2d$. The cost at 10 km is \$25.

For $d > 10$, retain that \$25 and add $1.5(d - 10)$. Thus $C(d) = 25 + 1.5(d - 10)$.

The two expressions agree at 10, but the rate changes from $\$\frac{2}{\text{km}}$ to $\$\frac{1.50}{\text{km}}$. Using $5 + 1.5d$ for the second part would incorrectly reprice the first 10 km.

$$C(d) = \begin{cases} 5 + 2d, & 0 \leq d \leq 10, \\ 25 + 1.5(d - 10), & d > 10. \end{cases}$$

CHECK 1.13.C5

Using the taxi model, find $C(12)$ and identify the rate after 10 km.

Use a continuous variable when partial kilometers are billed proportionally. If a business bills in complete kilometer increments, a step model or rounding rule would be needed instead. The assumption is part of the model.

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State assumptions precisely

Type of assumption	A useful statement
What stays consistent	The rate, unit price, geometry, or other mechanism is unchanged over the modeled interval.
How quantities vary	Dimensions scale together; an output changes linearly; a product remains constant.
Domain	Use physical restrictions, measured input range, nonzero denominators, and discrete counts.
Range	Require feasible outputs; interpret a continuous prediction before rounding a count.

EXAMPLE 1.13.7 | A prediction is not automatically a headcount

A model predicts 52.6 passengers for a bus with 50 seats. Explain the modeling issue.

Solution

A passenger count must be a nonnegative whole number. Reporting 52.6 as an actual count is inappropriate, although it may describe an expected demand.

Rounding to 53 does not solve the capacity issue. The prediction suggests excess demand or a model outside the bus-capacity constraint.

State whether the output is demand, expected attendance, or passengers actually carried. Those quantities can have different ranges.

Interpolation predicts within the observed input range. Extrapolation goes beyond it and depends more heavily on the model assumptions. Neither is guaranteed correct, but extrapolation needs particular scrutiny.

CHECK 1.13.C6

A model based on ages $12 - 18$ predicts performance at age 50. What is this called, and what needs justification?

A complete model choice

Name the family, cite evidence for its shape or rates, state the domain and range, and identify the assumptions that could fail.

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1.14 Function Model Construction and Application

FOCUS Construct models from conditions, transformations, data, and context. • Use linear, quadratic, cubic, and quartic regression appropriately. • Interpret predictions, rates, and inverse-proportional models with units.

Translate conditions into equations

After choosing a family, use the information in the problem to determine its parameters. A point (u, v) gives an equation $f(u) = v$. A zero a gives a factor $x - a$. A quadratic vertex (h, k) suggests $a(x - h)^2 + k$. Work from the representation that uses the given information most directly.

EXAMPLE 1.14.1 | A linear model through two points

A tank contains 180 L after 2 min and 120 L after 7 min. Assume a constant draining rate and construct $V(t)$.

Solution

The rate is $\frac{120-180}{7-2} = -12\text{L/min}$. Write $V(t) = -12t + b$.

Use $V(2) = 180$: $-24 + b = 180$, so $b = 204$. Thus $V(t) = 204 - 12t$.

If $t = 0$ marks the start and the draining pattern continues until empty, the physical domain is $0 \le t \le 17$ and range is $0 \le V \le 204$. The intercept 204 is inferred from the constant-rate assumption.

CHECK 1.14.C1

A linear model passes through $(1, 8)$ and $(4, 20)$. Find it.

Units in a model

In $V(t) = 204 - 12t$, 204 is measured in liters and -12 in liters per minute. Multiplying the rate by minutes gives a volume.

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Quadratics from three conditions

EXAMPLE 1.14.2 | Solve the coefficient system

A quadratic passes through $(0, 2)$, $(1, 3)$, and $(3, 17)$. Construct it.

Solution

Set $q(x) = ax^2 + bx + c$. The first point gives $c = 2$. The others give $a + b = 1$ and $9a + 3b = 15$.

Divide the second of these equations by 3 to get $3a + b = 5$. Subtract $a + b = 1$ to obtain $2a = 4$, so $a = 2$ and $b = -1$.

Thus $q(x) = 2x^2 - x + 2$. Checking gives $q(0) = 2$, $q(1) = 3$, and $q(3) = 17$.

EXAMPLE 1.14.3 | Use a vertex when it is given

A model has maximum height 12 m at $t = 2s$ and height 4 m at $t = 0$. Construct a quadratic height model.

Solution

Use $H(t) = a(t - 2)^2 + 12$. Substituting $t = 0$ gives $4 = 4a + 12$, so $a = -2$.

The model is $H(t) = -2(t - 2)^2 + 12$. On $t \geq 0$ it remains nonnegative through $t = 2 + \sqrt{6} \approx 4.449s$.

The negative-time root $2 - \sqrt{6}$ is outside the post-start domain. Context decides which root is relevant.

CHECK 1.14.C2

Construct a quadratic with zeros 1 and 5 and y-intercept 10.

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Polynomials from zeros, dimensions, and scale

EXAMPLE 1.14.4 | A least-degree quartic

A polynomial crosses at -2 and 3, touches at 1, and has $p(0) = 12$. Find the least-degree model.

Solution

Use $p(x) = a(x+2)(x-1)^2(x-3)$. The multiplicities $1, 2, 1$ sum to 4.

Substituting 0 gives $-6a = 12$, so $a = -2$.

The model is $p(x) = -2(x+2)(x-1)^2(x-3)$. Its leading coefficient is negative, so both ends fall. No derivative calculation is needed to verify the zeros and their local sign behavior.

A cubic geometric model can come directly from three linear dimensions, as with $V(x) = x(12-2x)(20-2x)$. At $x = 2\text{cm}$ this gives $V(2) = 256\text{cm}^3$. Dimensional reasoning helps check that the output units are cubic centimeters.

More generally, known zeros with required multiplicities give a product of factors. A nonzero value at another input determines the scale factor, provided the chosen product is nonzero there. Inconsistent conditions can make a model impossible.

CHECK 1.14.C3

Find the least-degree polynomial with simple zeros $-1, 2, 4$ and value $p(0) = 16$.

Check every condition

Substitute each stated point, verify every multiplicity, and check the domain. A model that matches one condition but violates another is not a solution.

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Regression uses the whole data set

A regression model chooses parameters to fit all supplied observations according to a criterion. Ordinary least squares minimizes the sum of squared vertical residuals, observed output minus predicted output. An exact model through selected points and a regression model serve different purposes.

x	-2	-1	0	1	2
Observed y	5	2	1	2	6

EXAMPLE 1.14.5 | A selected-point fit versus a quadratic regression

Fit a quadratic to the data using technology, and compare it with the curve through the three middle points.

Solution

The three middle points give $q(x) = x^2 + 1$. It predicts 5 at $x = 2$, while the observed output is 6.

Quadratic least squares using all five points gives $r(x) = \left(\frac{8}{7}\right)x^2 + \left(\frac{1}{5}\right)x + \frac{32}{35}$, approximately $1.142857x^2 + 0.2x + 0.914286$.

The fitted value at $x = 2$ is $\frac{206}{35} \approx 5.885714$, giving residual $6 - \frac{206}{35} = \frac{4}{35} \approx 0.114286$. The regression need not pass through any one chosen observation.

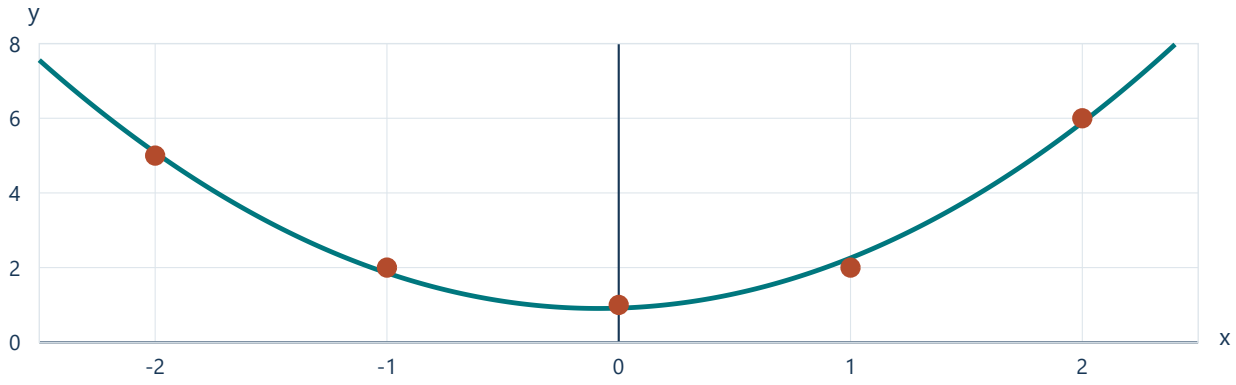


Figure 1.14.1. Observed points and the least-squares quadratic. Keep the data visible when judging a fit.

CHECK 1.14.C4

For the regression above, find the predicted output and residual at $x = 0$.

Keep full calculator precision for later calculations, then round reported values to a precision appropriate to the measurements. A visually close fit does not establish a causal explanation.

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A calculator workflow for four regression families

In the Desmos Graphing Calculator, enter data in a table with columns x_1 and y_1. Use the regression relation $\sim$, not an equals sign. The expressions below estimate parameters from the table. Use distinct parameter names or separate graphs when comparing models.

Family	Expression to enter
Linear	$y_1 \sim mx_1 + b$
Quadratic	$y_1 \sim ax_1^2 + bx_1 + c$
Cubic	$y_1 \sim ax_1^3 + bx_1^2 + cx_1 + d$
Quartic	$y_1 \sim ax_1^4 + bx_1^3 + cx_1^2 + dx_1 + e$

EXAMPLE 1.14.6 | Check cubic and quartic fits with exact data

Use cubic regression for $x = -2, -1, 0, 1, 2$ with $y = -7, 0, 1, 2, 9$. Use quartic regression for $x = 0, 1, 2, 3, 4, 5$ with $y = 1, 2, 17, 82, 257, 626$.

Solution

The cubic data are generated by $y = x^3 + 1$, so an exact least-squares fit has coefficients $1, 0, 0, 1$.

The quartic data are generated by $y = x^4 + 1$, so an exact fit has coefficients $1, 0, 0, 0, 1$. Technology may display tiny rounding errors near zero.

Both sets have more observations than the minimum needed for interpolation. Exact fits here are intentional checks of the workflow, not a claim that measured data normally follow perfect polynomials.

CHECK 1.14.C5

A quartic passes through five noisy points exactly. Is zero residual sufficient reason to prefer it over a quadratic?

For an actual investigation, graph the data and fitted curve, inspect representative residuals, and judge whether the proposed domain and shape make sense. More parameters can improve a numerical fit while weakening a model's usefulness.

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Construct a piecewise model

EXAMPLE 1.14.7 | Join two stages consistently

A tank starts with 20 L and fills at 6L/min for 5 min. It then drains at 4L/min until empty. Let t measure minutes since filling began.

Solution

During filling, $V(t) = 20 + 6t$ for $0 \le t \le 5$. At $t = 5$ the tank contains 50 L.

During draining, measure elapsed draining time by $t - 5$: $V(t) = 50 - 4(t - 5)$ for $5 < t \le 17.5$.

The two formulas meet at 50 L. Emptying takes $\frac{50}{4} = 12.5$ additional minutes, so the final time is 17.5 min. The range is $[0, 50]L$.

$$V(t) = \begin{cases} 20 + 6t, & 0 \le t \le 5, \\ 50 - 4(t - 5), & 5 < t \le 17.5. \end{cases}$$

The boundary belongs to only one displayed interval, so each input has one assigned rule. Here either rule would give the same value at $t = 5$, but a careful domain specification also works when a model has a jump.

CHECK 1.14.C6

Find $V(8)$ and the average rate on $[0, 8]$ for this tank.

A changing stage versus an average

The tank is draining at $t = 8$ even though its average rate over $[0, 8]$ is positive. A whole-interval rate need not describe the final stage.

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Inverse proportionality and inverse squares

If $xy = k$ is constant, y is inversely proportional to x: $y = \frac{k}{x}$. If $yx^2 = k$ is constant, $y = \frac{k}{x^2}$ is an inverse-square model. These are rational functions. The context usually supplies a positive-input domain.

EXAMPLE 1.14.8 | An inverse-square force model

A force magnitude $F(d)$ varies inversely with squared distance d. It is 18 N at 2 m. Construct the model and find the force at 3 m.

Solution

Write $F(d) = \frac{k}{d^2}$. The observation gives $18 = \frac{k}{4}$, so $k = 72\text{N} \cdot m^2$.

Thus $F(d) = \frac{72}{d^2}$ for $d > 0$. At 3m, $F(3) = \frac{72}{9} = 8\text{N}$.

Doubling distance divides the force by 4. The model assumes the underlying physical conditions represented by k remain fixed.

EXAMPLE 1.14.9 | Inverse variation with shared work

Twelve equal-speed pumps empty a reservoir in 5 h. Under ideal inverse proportionality, how long would 8 pumps take?

Solution

The product of number of pumps and time is $12 \cdot 5 = 60$ pump-hours. Thus $T(n) = \frac{60}{n}$ hours.

$T(8) = 7.5h$. The model assumes identical constant pump rates, no interference, and the same reservoir volume.

> **CHECK 1.14.C7**
>
> For $F(d) = \frac{72}{d^2}$, find the average rate of change from $d = 2$ to $d = 3\text{m}$.

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Apply, interpret, and test the model

A complete modeling response does more than provide an equation. Identify the quantities and units, describe why the chosen family is appropriate, solve for parameters, state restrictions, answer the question, and check the result against both data and context.

EXAMPLE 1.14.10 | Use a model's rate language accurately

For $F(d) = \frac{72}{d^2}$ with $d > 0$, describe what happens as d increases. Compare $[2, 3]$ and $[3, 4]$.

Solution

The force values are 18,8,4.5 N at distances 2,3,4 m. The average rates on the two intervals are -10 and -3.5N/m.

Force decreases, but the decrease becomes less rapid as distance increases. The graph is decreasing and concave up.

The comparison of two intervals supports the local interpretation. The full formula provides the model structure; two arbitrary data intervals alone would not prove global concavity.

Extrapolation must respect the mechanism. A cost model can fail when bulk discounts begin; a geometric model can fail when a dimension becomes nonpositive; an inverse-square approximation can fail when an idealized point-object assumption becomes inappropriate.

CHECK 1.14.C8

A fitted population model gives -8 people at a future input. Should this be reported as a valid prediction?

Unit 1 synthesis

Use representations together: a formula explains structure, a table supports numerical comparisons, a graph reveals shape, and a verbal statement connects the mathematics to the modeled quantity. Agreement across them is a strong error check.

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Choose your next move

The question asks for...	Start with...
A function value	A legal input, then substitution or graph/table reading.
Average rate	Endpoint output change divided by input change, with units.
Concavity	Whether signed rates increase or decrease.
Zeros	Factors and the original domain.
End behavior	Degree and leading term, or a leading-term ratio.
A vertical asymptote	A denominator factor that remains after cancellation.
A hole	An original excluded input and the reduced expression value there.
A slant equation	Polynomial long division.
A transformation	Solve the new inside expression = the old input.
A model	Context and family first; conditions or regression second.

Three statements to keep separate

The output is positive. The output is increasing. The rate is increasing. These describe sign, direction, and concavity, respectively; none is a substitute for another.

When an answer seems plausible, check the original problem. Confirm domain restrictions, endpoint inclusion, exact versus approximate values, and units. Test a claimed algebraic identity at a legal input or expand it back.

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One rational function, several questions

$$r(x) = \frac{(x+2)(x-1)^2}{(x-1)(x-3)}$$

The original domain excludes 1 and 3. On that domain, $r(x) = \frac{(x+2)(x-1)}{x-3}$. Use the original expression to keep the domain, and the reduced expression to describe the nearby behavior.

Feature	Conclusion and reason
Zero	$x = -2$ is legal and makes the numerator zero. $x = 1$ is excluded.
Hole	$(1, 0)$, obtained from the reduced expression.
Vertical asymptote	$x = 3$; a denominator factor remains.
One-sided limits at 3	Left: $-\infty$. Right: $+\infty$. The reduced numerator stays positive nearby.
Slant asymptote	$y = x + 4$, since $r(x) = x + 4 + \frac{10}{x-3}$.
End limits	Left: $-\infty$. Right: $+\infty$. The linear quotient dominates.
y-intercept	$(0, \frac{2}{3})$.

Check the division by multiplying $(x-3)(x+4) + 10 = x^2 + x - 2 = (x+2)(x-1)$. A graph may hide the hole, but the algebra identifies it exactly.

Bridge to Calculus

Unit 1 connects a function value, a limiting value, and a rate. Preserve those distinctions when later studying continuity, derivatives, and accumulation.

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