Calculus AB
Student reference

Use each lesson to connect a calculation to a picture, a definition, or a quantity you can explain.

Study a topic together

Each home lesson opens with an overview and two examples. Its numbered CED topics then show how to handle distinct situations. Work a model yourself before reading the steps. Use the assigned C checks without looking at the solutions. Some topics address several objectives and have more than two checks.

Choose the right amount of practice

Follow the weekly assignment strip selected with your tutor. Use the assigned model, C and P numbers for each objective. A familiar topic may need only a fresh independent probe. For a missed skill, work a model, try its checks, and save a different problem for a later day. The remaining pages support repairs and fuller study.

Check your work

The check key follows the last lesson. The same topic number identifies the matching workbook practice and tutor solutions. The formula reference is at the back.

> **First step.** Use your completed diagnostic and feedback. Begin with fresh domain and function-subtraction attempts selected by your tutor. Study only the repair those attempts show you need.

Editorial revision 1.1.0 • November 12, 2026

LESSON DIRECTORY

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LESSON 01

Algebra, radians, and geometry

Simplify an expression without losing its domain, read exact trig values, and translate a picture into an equation.

College Board prerequisites, p. 7 • OpenStax Calculus Vol. 1, sections 1.1–1.5

The idea

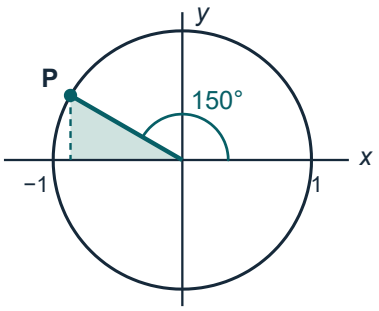
Before simplifying a fraction, identify every value that makes its denominator zero. Factor, then cancel common factors. Two formulas may agree at every shared input even though one has a hole in its graph.

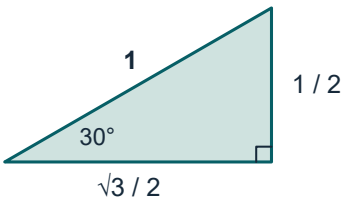
On the unit circle, the point at angle θ is $(\cos\theta, \sin\theta)$. Start at $(1, 0)$ and rotate counterclockwise for positive angles. Use the acute reference angle for size and the quadrant for sign. Where $\cos\theta \neq 0$, $\tan\theta = \sin\theta / \cos\theta$.

One radian subtends an arc one radius long. For a circle of radius r and an angular sweep of magnitude $\theta \geq 0$ radians, arc length is $s = r\theta$ and sector area is $A = r^2\theta/2$. For a signed rotation, use its magnitude to find length. A full turn is 2π radians or 360°. Calculus trig rules use radians.

Name quantities before choosing a geometry formula. A cylinder of radius r and height h has volume $\pi r^2 h$. If two quantities both change, an additional relationship, often from similar triangles or a fixed constraint, can let you substitute for one of them.

A composition applies its inner function first. For $f(u) = \sqrt{u}$ and $g(x) = x - 2$, $(f \circ g)(x) = \sqrt{x-2}$ requires $x \geq 2$, while $(g \circ f)(x) = \sqrt{x} - 2$ requires $x \geq 0$. Exponentials and logs undo each other. For $b > 0$ and $b \neq 1$, $b^v = u$ means $\log_b u = v$, with $u > 0$. Thus $e^{2x} = 5$ gives $x = (\ln 5)/2$.





Reference triangle

The terminal point P is on the unit circle at 150°. The enlarged reference triangle shows the side lengths.

What are the signed coordinates of P? Use them to find the tangent of 150°.

LESSON 01 / WORKED EXAMPLES

Algebra, radians, and geometry

Example 1

A function is given by $f(x) = \left(x^2 - 9\right)/(x - 3)$. What line describes its graph, and which point is missing?

1. Factor the numerator as $(x - 3)(x + 3)$. The original denominator requires $x \neq 3$.
2. For every permitted input, $f(x) = x + 3$. At $x = 3$ that line would have height 6. Draw the line with an open circle at $(3, 6)$.

Keeping track of that one missing input will let you distinguish a function value from a limit.

Example 2

A wheel of radius 12 cm rotates through 150°. Find the distance traveled by a mark on its rim and its ending coordinates relative to the center if it began at $(12, 0)$ and rotated counterclockwise.

1. In radians the angle is $150\pi/180 = 5\pi/6$. The mark travels $12\left(5\pi/6\right) = 10\pi$ cm.
2. The reference angle is $\pi/6$. The ending point lies in quadrant II, so its coordinates are $12\left(-\sqrt{3}/2, 1/2\right) = \left(-6\sqrt{3}, 6\right)$ cm.

Arc length measures travel along the circle. The coordinates describe the final location.

> **Common error.** Cancel factors only. For example, $(x + 4)/x = 1 + 4/x$ for $x \neq 0$. Removing the two appearances of x would change almost every output.

Explain it aloud

Show how you would recover the exact sine and cosine of $7\pi/6$ from a sketch if you forgot both values.

LESSON 02

Limits and continuity

Use a graph, a table, algebra, or a bound to justify a limit, then decide what the result implies about continuity.

CED 1.1–1.16 • OpenStax Calculus Vol. 1, sections 2.1–2.4, 4.6

The idea

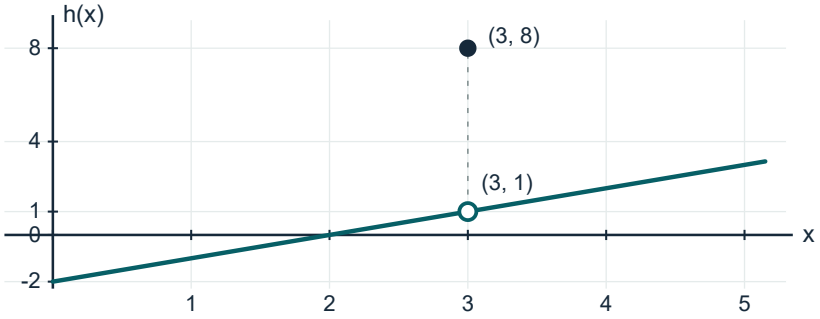
A two-sided limit concerns values near a point. Read a table in input order. At inputs 1.90, 1.99, 2.00, 2.01, 2.10, suppose the outputs are 3.80, 3.98, 7.00, 4.02, 4.20. From each side of 2 the values suggest 4, whereas $f(2) = 7$. A finite two-sided limit requires the two finite one-sided limits to agree.

Sum and product limits are the sum and product of the finite component limits. Quotients work when the limiting denominator is nonzero. Thus if $f \to 2$ and $g \to 3$, $(f+g)/g \to 5/3$. If $f \to L$ and h is continuous at L, then $h(f) \to h(L)$. When substitution instead gives $0/0$, try factoring or multiplying by a conjugate.

If $g(x) \le f(x) \le h(x)$ near a and both bounds approach L, then $f(x) \to L$. This squeeze argument works for a bounded oscillation times a factor tending to zero. A standard trig limit in radians is $\lim_{u \to 0} (\sin u)/u = 1$. With $u = 3x$, $(\sin 3x)/x = 3(\sin u)/u \to 3$.

Continuity at a requires a defined $f(a)$, an existing finite limit, and equality between them. At endpoints of a closed interval use the limit from inside the interval. The Intermediate Value Theorem (IVT) guarantees that a function continuous on $[a, b]$ attains every value between $f(a)$ and $f(b)$.

A vertical asymptote at $x = a$ occurs when at least one one-sided limit is infinite. A horizontal asymptote $y = L$ comes from a finite limit as $x \to \infty$ or $x \to -\infty$. For a rational function at infinity, dividing numerator and denominator by the highest denominator power often reveals its end behavior.



The curve has an open circle at (3, 1) and a filled point at (3, 8).

Read both one-sided limits at 3 and the function value. Which single point would you move to make h continuous?

LESSON 02 / WORKED EXAMPLES

Limits and continuity

Example 1

Can a choice of k make the following function continuous at 2? For $x < 2$ use $f(x) = \left(x^2 - 4\right)/(x - 2)$, and for $x \geq 2$ use $f(x) = kx + 1$.

1. On the left, cancellation gives $x + 2$, so the left-hand limit is 4.
2. On the right, both the limit and the function value are $2k + 1$. Set $2k + 1 = 4$, so $k = 3/2$.

Matching the two approaches alone would overlook the point value if a separate rule defined it.

Example 2

A continuous calibration function has $C(1) = 7.1$ and $C(3) = 8.9$. What can you conclude about $C(t) = 8$?

1. The number 8 lies strictly between the endpoint values.
2. Continuity on $[1, 3]$ lets us apply the IVT. At least one t in $(1, 3)$ satisfies $C(t) = 8$.

The data give existence. Several inputs could yield 8, and the theorem does not locate them.

> **Common error.** A short table can suggest a limit. It cannot prove one for an otherwise unspecified function because the behavior between the sampled inputs is unknown.

Explain it aloud

A graph has an open circle at $(1, 4)$ and a filled point at $(1, -2)$. Describe what else you must inspect to decide its two-sided limit at 1.

LESSON 02 / CED 1.1

Change at an instant

Interpret an instantaneous rate as the limit of average rates over shrinking time intervals.

CHA-1.A • OpenStax Calculus Volume 1, §2.1, A Preview of Calculus

How fast is a ball moving when the clock reads exactly two seconds? A stopwatch measures a time interval. At an isolated instant the elapsed time is zero, so dividing distance by elapsed time gives no rate. Instead compare positions at the target time and a nearby time. For position $s(t)$, the average velocity from $t = a$ to $t = a + h$ is $[s(a + h) - s(a)]/h$ for nonzero h. Negative h uses an earlier measurement. If these slopes approach a common number from both sides as h shrinks, that number is the instantaneous velocity. Other changing quantities use the same idea with their own units.

Worked example 1

A ball's position is $s(t) = t^2 + 3t$ meters. Find the average velocities from 2 s to 2.5 s and from 2 s to 2.1 s. What do they suggest about velocity at 2 s?

1. At 2 s the position is 10 m. At 2.5 s it is 13.75 m, giving $(13.75 - 10)/0.5 = 7.5$ m/s. At 2.1 s it is 10.71 m, giving 7.1 m/s.
2. For any small nonzero h, $s(2 + h) = 10 + 7h + h^2$. The average is $7 + h$, which tends to 7 from either side. Instantaneous velocity is 7 m/s.

Worked example 2

Tank volume is 42.40, 42.00, and 41.56 L at 4.8, 5.0, and 5.2 min. Estimate its instantaneous rate at 5 min.

1. The left and right average rates are $(42.00 - 42.40)/0.2 = -2.0$ and $(41.56 - 42.00)/0.2 = -2.2$ L/min.
2. Across the centered interval the average is $-0.84/0.4 = -2.1$ L/min. Use this as an estimate. The three measurements do not fix the instantaneous rate exactly.

Check your understanding

1.1 C 1 Why does $[s(5) - s(5)]/(5 - 5)$ fail to give the velocity at 5?

1.1 C 2 Over shrinking intervals ending and starting at $t = 2$, the left average rate tends to -3 and the right average rate tends to 4. Is there one two-sided instantaneous rate?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.2

Reading and writing limit notation

Translate a limiting statement into symbols and explain exactly which inputs and outputs it describes.

LIM-1.A / LIM-1.B • OpenStax Calculus Volume 1, §2.2, The Limit of a Function

In $\lim_{x \to a} f(x) = L$, a is an input being approached and L is the approached output. The assertion means you can make $f(x)$ stay as close to L as you wish by taking x sufficiently close to a, while keeping $x \neq a$. Approach from both sides unless a superscript specifies one. In $x \to a^-$, inputs are smaller than a. In $x \to a^+$, they are larger. The signs tell you where the inputs are, regardless of whether the outputs rise or fall. A limit statement by itself gives no value at the target input.

Worked example 1

A runner's speed $v(t)$ tends to 6 m/s as time approaches 12 s from below. Write this in symbols and say whether it determines $v(12)$.

1. The input approaches 12 through earlier times. Write $\lim_{t \to 12^-} v(t) = 6$.
2. This describes readings before 12 s. It gives neither the exact speed at 12 s nor the behavior just afterward.

Worked example 2

Read $\lim_{x \to -2} p(x) = 4$ aloud. Could it hold if $p(-2) = 0$?

1. As x approaches -2 from either side, $p(x)$ approaches 4. For example, $x = -2.001$ is on the left and $x = -1.999$ is on the right.
2. Yes. Define $p(x) = x + 6$ except at $x = -2$, and set $p(-2) = 0$. The nearby values still approach 4.

Worked example 3

On either side of $x = 4$, a graph approaches the open point $(4, -2)$. Its filled point is $(4, 6)$. Record the two-sided limit and interpret what each number in the statement names.

1. Approaching from either direction with $x \neq 4$, the graph heights approach -2. Write $\lim_{x \to 4} f(x) = -2$.
2. The 4 identifies the target input; -2 identifies the approached output. The separate statement $f(4) = 6$ gives the actual height at the input.

Check your understanding

1.2 C 1 Translate “as u approaches 3 through larger values, $P(u)$ approaches 18.” Does this statement specify $P(3)$ or what happens as u approaches 3 from below?

1.2 C 2 Read $\lim_{x \to 0} g(x) = -1$ in words, then write its two one-sided limit statements. Must the outputs approach -1 through smaller values?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.3

Estimating limits from graphs

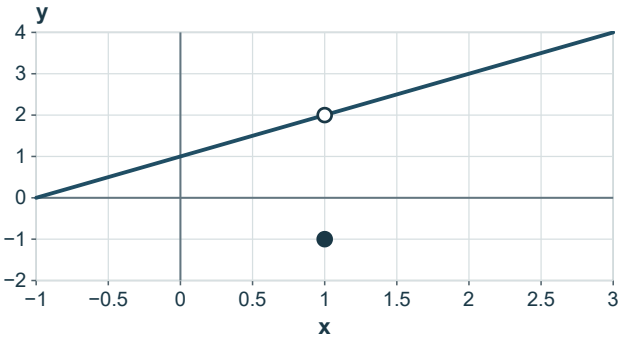
Read approaching heights independently on both sides of an input and recognize when a finite limit fails.

LIM-1.C • OpenStax Calculus Volume 1, §2.2, The Limit of a Function

Place a finger on each branch near the target x and trace toward the target. Where does each branch head vertically? An open circle may mark a missing value the branch approaches. A filled point gives the value at the input itself. If both approaching heights agree, that height is the two-sided limit. A jump, unbounded growth, or persistent oscillation can prevent a finite limit. A calculator graph samples points. A hole smaller than a pixel or an oscillation faster than the pixel spacing may be invisible. Test apparent features with a second representation.

Worked example 1

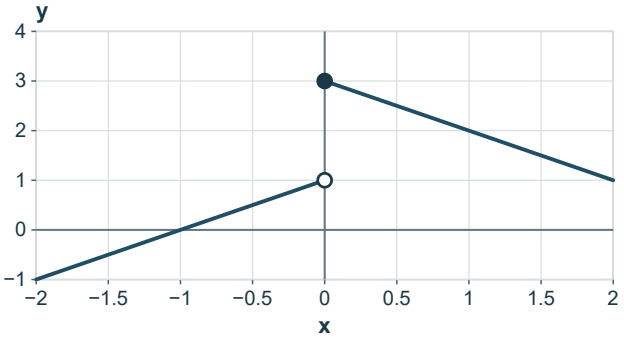
Read both one-sided limits and $f(1)$ from the graph.



1. The branch on each side approaches the open point $(1, 2)$. Thus $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$.
2. At input 1 the filled point is at height -1. The two-sided limit is 2 and $f(1) = -1$.

Worked example 2

Read the behavior at $x = 0$. Which side, if either, agrees with $g(0)$?



1. Tracing from the left gives height 1. Tracing from the right gives height 3.
2. The filled point at height 3 gives $g(0) = 3$, which agrees with the right-hand limit. The unequal one-sided limits rule out a two-sided limit.

LESSON 02 / CED 1.3 / CONTINUED

Estimating limits from graphs

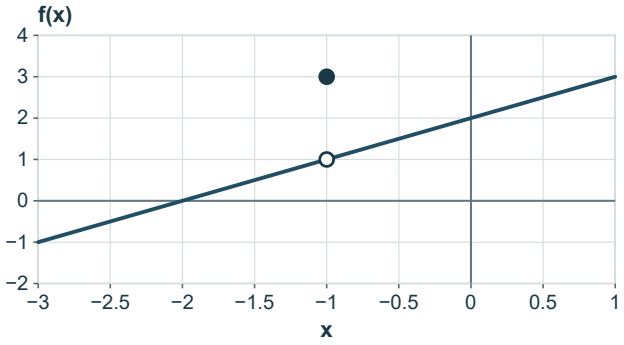
Read approaching heights independently on both sides of an input and recognize when a finite limit fails.

LIM-1.C • OpenStax Calculus Volume 1, §2.2, The Limit of a Function

Check your understanding

1.3 C 1 For $x \neq 0$, a curve oscillates between -1 and 1 and keeps attaining both values arbitrarily close to 0. Can it have a finite limit there?

1.3 C 2 Read the graph at $x = -1$. State the two one-sided limits, the two-sided limit, and the function value. What detail might a low-resolution graph conceal?



Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.4

Estimating limits from tables

Choose nearby inputs on both sides, compare numerical trends, and distinguish an estimate from an exact limit.

LIM-1.C • OpenStax Calculus Volume 1, §2.2, The Limit of a Function

First find the target input in the table. On the left, read rows toward the target from smaller inputs. On the right, read toward it from larger inputs, which often means reading backward. Compare the two output trends. A row at the target, if present, reports the function value. A finite table can supply an estimate. Points closer to the target may reveal behavior hidden between the sampled rows, and rounded displays can hide small changes.

Worked example 1

Estimate the limit at 3 and report the point value.

x	2.9	2.99	3	3.01	3.1
$f(x)$	6.8	6.98	1	7.02	7.2

1. From the left, 6.8 then 6.98 approach 7. From the right, 7.2 then 7.02 also approach 7.
2. The data suggest $\lim_{x \to 3} f(x) \approx 7$. The middle entry separately gives $f(3) = 1$.

Worked example 2

Estimate each one-sided limit at 0.

x	-0.1	-0.01	0	0.01	0.1
$f(x)$	1.90	1.99	4	5.01	5.10

1. Negative inputs suggest a left limit of 2. Positive inputs suggest a right limit of 5.
2. These trends suggest a jump. With only the given samples, we cannot prove what happens at every closer input.

Check your understanding

1.4 C 1 A table gives $f(3.9) = 1.8$, $f(3.99) = 1.98$, $f(4.01) = 4.02$, and $f(4.1) = 4.2$. Estimate the one-sided limits at 4. What do the samples suggest about the two-sided limit?

1.4 C 2 If every sampled value rounded to two decimal places is 1.00, is the exact limit necessarily 1?
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.5

Using limit laws

Build a limit from simpler limits and verify the condition for each operation.

LIM-1.D • OpenStax Calculus Volume 1, §§2.3–2.4, The Limit Laws and Continuity

Suppose $f(x) \to A$ and $g(x) \to B$ as $x \to a$, with both limits finite. Addition, subtraction, multiplication, and constant multiples pass through the limit. Division gives A/B when $B \neq 0$. For a composition $H\big(g(x)\big)$, if H is continuous at B, its limiting value is $H(B)$. These rules also hold for one-sided approaches. In a piecewise formula, choose the expression on the side through which x travels. Its rule at the isolated target has no bearing on that side's limit.

Worked example 1

As $x \to 4$, suppose $f(x) \to 3$ and $g(x) \to -2$. Evaluate $\lim_{x \to 4} \dfrac{f(x)^2 + 2g(x)}{f(x) - g(x)}$.

1. Use product and sum laws in the numerator to get $3^2 + 2(-2) = 5$. The difference in the denominator tends to $3 - (-2) = 5$.
2. The limiting denominator is nonzero, so the quotient law gives $5/5 = 1$.

Worked example 2

Let $q(x) = x + 3$ for $x < 1$ and $q(x) = x^2$ for $x \geq 1$. Find each one-sided limit of $\sqrt{q(x)}$ at 1.

1. On the left, $q(x) \to 4$. On the right, $q(x) \to 1$.
2. The square root is continuous at each of those positive values. The left composite limit is 2 and the right is 1. There is no two-sided composite limit.

Check your understanding

1.5 C 1 If $p(x) \to 4$ and $q(x) \to 0$, what can the quotient law alone conclude about $p(x)/q(x)$?

1.5 C 2 If $g(x) \to 0$, determine $\lim_{x \to a} \cos\big(g(x)\big)$ and name the condition that justifies substitution.
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.6

Rewriting expressions before taking limits

Remove an algebraic obstacle using a factor, conjugate, or trig identity, then take the limit.

LIM-1.E • OpenStax Calculus Volume 1, §2.3, The Limit Laws

Substituting the target in a quotient can produce $0/0$, which leaves its limiting behavior undecided. Work at nearby inputs where the original expression is defined. Factoring can expose a common factor, and a conjugate converts a difference of square roots into a simpler difference. Trig identities can also expose a factor. After rewriting, keep the limit notation until the simplified expression can be evaluated. Canceling a factor preserves the nearby outputs and leaves the original function's domain unchanged.

Worked example 1

Evaluate $\lim_{x \to -2} \frac{x^2 + x - 2}{x + 2}$.

1. Substitution gives $0/0$. Factor $x^2 + x - 2 = (x + 2)(x - 1)$.
2. At nearby inputs $x \neq -2$ the quotient equals $x - 1$. Its limit is $-2 - 1 = -3$.

Worked example 2

Evaluate $\lim_{h \to 0} \frac{\sqrt{9 + 2h} - 3}{h}$.

1. Multiply numerator and denominator by $\sqrt{9 + 2h} + 3$. The numerator becomes $(9 + 2h) - 9 = 2h$.
2. For $h \neq 0$, cancellation gives $2/\left(\sqrt{9 + 2h} + 3\right)$. Its limit is $2/(3 + 3) = 1/3$.

Check your understanding

1.6 C 1 Use a trig identity to evaluate $\lim_{x \to 0} \frac{1 - \cos^2 x}{\sin x}$.

1.6 C 2 A student divides both numerator and denominator of $\left(x^2 + 2\right)/(x + 2)$ by "the $+2$." Why is this invalid?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.7

Choosing a limit procedure

Use the presentation and the result of initial substitution to choose and justify a method.

Skill 1.C • OpenStax Calculus Volume 1, §§2.2–2.3, The Limit of a Function and The Limit Laws

Read the representation before calculating. A graph calls for tracing each side and a table calls for a trend estimate. With a formula, check the domain and try direct substitution when continuity permits. A nonzero denominator often finishes the problem. When the result has the form $0/0$, look for equivalent expressions. A nonzero numerator over a denominator approaching zero calls for sign analysis. Different formulas on opposite sides call for two one-sided calculations. Name the feature that makes your chosen method suitable.

Worked example 1

Choose methods for $\lim_{x \to 2} \left(x^2 + 1\right) / (x + 1)$ and $\lim_{x \to 2} \left(x^2 - 4\right) / (x - 2)$. Why should the methods differ?

1. In the first problem the limiting denominator is 3. Direct substitution gives $5/3$.
2. In the second, substitution gives $0/0$. Factor $x^2 - 4 = (x - 2)(x + 2)$ and cancel for $x \neq 2$. The simplified expression tends to 4.

Worked example 2

Compare $\lim_{h \to 0} \left(\sqrt{4 + h} - 2\right)/h$ and $\lim_{h \to 0} |h|/h$. Both give $0/0$ on substitution.

1. The radical difference suggests a conjugate. For $h \neq 0$ the first quotient equals $1/\left(\sqrt{4 + h} + 2\right)$, with limit $1/4$.
2. The absolute value changes formula at zero. The second quotient is -1 for $h < 0$ and 1 for $h > 0$, so no two-sided limit exists.

Check your understanding

1.7 C 1 You have a graph with branches on both sides of $x = 2$ and no formula. What is a defensible first move?

1.7 C 2 For $\lim_{x \to 3} (x + 3) / (x - 3)$, would factoring to remove a hole be justified?
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.8

Using the squeeze theorem

Find bounds near the target, verify a common limiting value, and use it to establish the inner limit.

LIM-1.E • OpenStax Calculus Volume 1, §2.3, The Limit Laws; §3.5, Derivatives of Trigonometric Functions (opening trig limits)

Suppose $g(x) \le f(x) \le h(x)$ holds at every sufficiently nearby input other than the target. If g and h both approach the same number, f must approach it too. For an oscillating factor with $|\sin u| \le 1$ or $|\cos u| \le 1$, an absolute-value bound often supplies both bounds at once. If you multiply an inequality by a quantity whose sign can change, use its absolute value to avoid reversing an inequality incorrectly.

Worked example 1

Find $\lim_{x \to 0} x \cos\left(1/x\right)$ without assuming cosine has a limit here.

1. For $x \neq 0$, $\left|x \cos\left(1/x\right)\right| \le |x|$. Thus $-|x| \le x \cos\left(1/x\right) \le |x|$.
2. The lower and upper bounds both tend to 0. The squeeze theorem gives the limit 0.

Worked example 2

For small positive u in radians, unit-circle geometry gives $\sin u \le u \le \tan u$. Use this to establish $\lim_{u \to 0} \sin u/u$.

1. For $0 < u < \pi/2$, we have $u > 0$ and $\cos u > 0$. Divide $\sin u \le u$ by u to obtain $\sin u/u \le 1$. Rewrite $u \le \tan u$ as $u \cos u \le \sin u$, then divide by u to get $\cos u \le \sin u/u$.
2. As $u \to 0^+$, both bounds tend to 1, so the ratio tends to 1. The ratio is even because $\sin(-u) = -\sin u$. The left limit also equals 1.
3. This result uses radians. It also gives $(1 - \cos u)/u = 2 \sin^2\left(u/2\right)/u = \left[\sin\left(u/2\right)/\left(u/2\right)\right] \sin\left(u/2\right) \to 0$.

Check your understanding

1.8 C 1 If $2 - x^2 \le p(x) \le 2 + |x|$ near zero, what is $\lim_{x \to 0} p(x)$?

1.8 C 2 Does $-1 \le \sin\left(1/x\right) \le 1$ prove a limit as $x \to 0$?
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.9

Connecting representations of limits

Translate between a formula, a table, a graph, and a statement about nearby inputs.

Skill 2.C • OpenStax Calculus Volume 1, §§2.2–2.3, The Limit of a Function and The Limit Laws

Start with three questions that work in every representation. What input are we approaching? What does each side do? What happens at the target itself? Converting a graphic into a table or a formula into a sketch tests whether you have kept those facts separate. When representations disagree, check the domain, whether the graph uses open or filled points, and whether the table samples both sides. Exact algebra can settle a conjecture suggested by a few plotted or tabulated values.

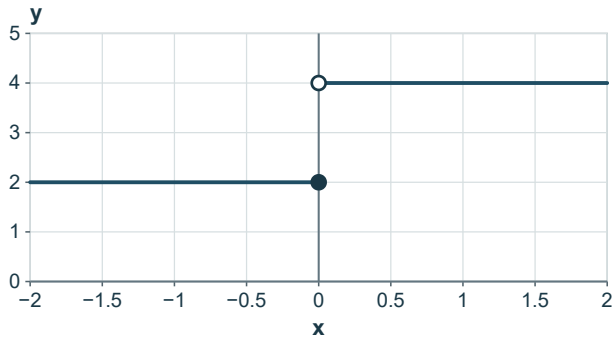
Worked example 1

A table for $F(x) = \left(x^2 - 16\right)/(x - 4)$ lists $F(3.99) = 7.99$ and $F(4.01) = 8.01$. Describe the graph and state the limit.

1. Factor $x^2 - 16 = (x - 4)(x + 4)$. Thus the graph is the line $y = x + 4$ with its point at $x = 4$ removed.
2. Both nearby table entries lie on that line. Algebra confirms $\lim_{x \to 4} F(x) = 8$. The missing point is $(4, 8)$.

Worked example 2

Turn the displayed graph into three symbolic statements, then use it to supply the values at $x = -0.1$ and $x = 0.1$.



1. The left branch is horizontal at 2 and the right branch at 4. Thus $\lim_{x \to 0^-} g(x) = 2$ and $\lim_{x \to 0^+} g(x) = 4$. The filled point gives $g(0) = 2$.
2. The branch values give $g(-0.1) = 2$ and $g(0.1) = 4$. A table using only negative inputs would miss the jump.

Check your understanding

1.9 C 1 A line passes through $(2, 5)$ with an open point there; the function value is 9. If the same line approaches from either side, write the two relevant facts in symbols.

1.9 C 2 At $x = 0.99$ and $x = 1.01$ a table gives outputs 4.99 and 5.01. Could both $f(1) = 5$ and $f(1) = 8$ be consistent with the table?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.10

Recognizing types of discontinuity

Classify a discontinuity from its one-sided behavior and determine whether changing one point could repair it.

LIM-2.A • OpenStax Calculus Volume 1, §2.4, Continuity

At an interior input, compare both one-sided limits first. If they meet at a finite height but the point is missing or misplaced, the discontinuity is removable. Different finite one-sided limits create a jump. If the output tends to positive or negative infinity from at least one side, there is an infinite discontinuity with a vertical asymptote. These categories do not exhaust all behavior: a bounded function can oscillate indefinitely near a point. Only a finite common limit lets one replacement value restore continuity.

Worked example 1

The expression $f(x) = \left(x^2 - 1\right)/[\,(x-1)\,(x-3)\,]$ is undefined at 1 and at 3. Classify both breaks.

1. Away from those inputs the formula is $(x+1)/(x-3)$. At 1 it tends to $2/(-2) = -1$, so the missing point is removable.
2. At 3 the simplified numerator stays positive while the denominator approaches zero through opposite signs. The one-sided limits are $-\infty$ and $+\infty$, producing an infinite discontinuity.

Worked example 2

Near $t = 0$, a sensor records $S(t) = 2 + t$ on the left and $S(t) = 1 - t$ on the right. It records $S(0) = 2$. Classify the break.

1. Compute the left limit 2 and the right limit 1. These heights are finite and unequal.
2. The break is a jump. Moving the point at 0 may match one branch, but cannot make the two approaching values agree.

Check your understanding

1.10 C 1 A function has a filled point at $(4, 0)$ and both one-sided limits at 4 equal 0. Should this point be classified as a removable discontinuity?

1.10 C 2 For $x \neq 0$, $g(x) = \sin\left(1/x\right)$, and $g(0) = 0$. Is the discontinuity at zero infinite or removable?
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.11

Testing continuity at one point

Check the point value, the two-sided limit, and their equality in a written continuity argument.

LIM-2.A • OpenStax Calculus Volume 1, §2.4, Continuity

At an interior point a, check three facts in order. First, find the assigned value $f(a)$. Second, compute or establish the finite two-sided limit as $x \to a$. Finally, compare the two. When both exist and are equal, you have justified continuity at a. If a condition fails, identify it and show the evidence. For a piecewise definition, the branch that contains the equality sign gives $f(a)$, while all nearby branches can affect the limit.

Worked example 1

For $x < 1$ let $h(x) = 2x + 1$. For $x \geq 1$ let $h(x) = x^2 + 2$. Is h continuous at 1?

1. The second branch gives $h(1) = 1^2 + 2 = 3$. The left limit from $2x + 1$ is 3 and the right limit from $x^2 + 2$ is 3.
2. Thus the two-sided limit exists and equals $h(1)$, so h is continuous at 1.

Worked example 2

For $x \neq -2$ let $g(x) = x^2$ and set $g(-2) = 1$. Determine exactly which part of the continuity test fails.

1. The assigned value exists and equals 1. The nearby values are given by the polynomial x^2, so their two-sided limit is 4.
2. The comparison $4 \neq 1$ fails. The function has a removable discontinuity at -2.

Check your understanding

1.11 C 1 Suppose $p(5) = 2$ and $\lim_{x \to 5^-} p(x) = 2$. Which further fact is needed for continuity at this interior input?

1.11 C 2 A function is undefined at $x = 1$ but its limit there is finite. Can it be continuous at 1 as currently defined?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.12

Continuity on intervals

Find intervals of continuity from domains, piece boundaries, and one-sided endpoint limits.

LIM-2.B • OpenStax Calculus Volume 1, §2.4, Continuity

A function is continuous throughout an open interval if it is continuous at every input there. For a closed endpoint, require the limit from within the interval to equal the endpoint value. Polynomials and exponentials are continuous for every real input. Rational functions require nonzero denominators. Trig, logarithmic, and power functions are continuous at inputs in their respective domains. For combinations, intersect all restrictions and check any boundary where the formula changes. List separate intervals instead of crossing an excluded input.

Worked example 1

List the maximal intervals of continuity for $F(x) = \sqrt{5-x}/(x-1)$.

1. The square root requires $x \leq 5$. The denominator excludes 1. On all remaining inputs the components are continuous.
2. The intervals are $(-\infty, 1)$ and $(1, 5]$. At 5, the limit from the left is $0 = F(5)$, allowing the closed endpoint.

Worked example 2

On $[0, 3]$, let $p(t) = t^2$ for $t < 2$ and $p(t) = 2t$ for $t \geq 2$. Check continuity on the entire interval.

1. Each polynomial branch is continuous on its interior. At 2 both branches tend to 4, which equals $p(2)$.
2. At 0 use the right limit 0, and at 3 use the left limit 6. These equal the endpoint values, so p is continuous on $[0, 3]$.

Check your understanding

1.12 C 1 Where is $\ln(x-2)$ continuous? Explain why the endpoint is included or excluded.

1.12 C 2 Give the maximal interval containing 0 on which $\tan x$ is continuous.
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.13

Repairing discontinuities

Choose point values or parameters that join both branches, and recognize breaks that a point value cannot fix.

LIM-2.C • OpenStax Calculus Volume 1, §2.4, Continuity

To repair a single missing or misplaced value, find the finite common limit and assign that number at the target. To choose parameters in a piecewise formula, write separate expressions for the left limit, right limit, and assigned value. At each joining input these must all agree. Two boundaries can give two equations for two unknowns. Check the resulting values in the original branches. If the two sides disagree for every allowable choice, no assignment at that point repairs the break.

Worked example 1

For $x \neq 4$ let $f(x) = \left(x^2 - 16\right) / (x - 4)$. Determine the unique choice of $f(4)$ that extends it continuously.

1. Cancel a factor for nearby $x \neq 4$ to obtain $f(x) = x + 4$. Its two-sided limit is 8.
2. Continuity requires $f(4) = 8$. Any other assignment would leave a mismatch with the same limit.

Worked example 2

Let $g(x) = 1$ for $x \leq 0$, $g(x) = ax + b$ for $0 < x < 2$, and $g(x) = 7$ for $x \geq 2$. Select a, b for continuity everywhere.

1. At 0 the middle expression must approach 1, giving $b = 1$.
2. At 2 the middle expression must approach 7, so $2a + b = 7$ and $a = 3$. The middle line $3x + 1$ now meets both endpoint values.

Check your understanding

1.13 C 1 For $x \neq 0$ let $p(x) = |x|/x$. Can assigning $p(0)$ create continuity there?

1.13 C 2 At $x = 2$ the left and right limits of q equal $3k - 1$ and $k + 5$. What are the required k and point value?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.14

Infinite limits and vertical asymptotes

Use one-sided signs to describe unbounded outputs and locate a vertical asymptote.

LIM-2.D • OpenStax Calculus Volume 1, §2.2, The Limit of a Function; §4.6, Limits at Infinity and Asymptotes

When x approaches a finite input a, its outputs may grow beyond every fixed upper bound or fall below every fixed lower bound. Write $+\infty$ or $-\infty$ to describe that behavior. Check the left and right sides separately. A line $x = a$ is a vertical asymptote if at least one such one-sided infinite limit exists. A two-sided infinite limit requires both sides to diverge with the same sign. A vertical asymptote can coexist with an assigned function value at a.

Worked example 1

For $f(x) = (x+1)/(x-2)$ determine the one-sided limits at 2 and identify an asymptote.

1. Near 2 the numerator is positive and near 3. The denominator is negative on the left and positive on the right, with magnitude shrinking to zero.
2. Hence $\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = +\infty$. The vertical asymptote is $x = 2$. The opposite signs prevent a two-sided infinite limit.

Worked example 2

A function equals $-4/(x+1)^2$ for $x \neq -1$ and has value $h(-1) = 6$. What happens near -1?

1. The squared denominator stays positive and approaches zero from both sides. A negative numerator then makes both sides tend to $-\infty$.
2. Write $\lim_{x \to -1} h(x) = -\infty$. The line $x = -1$ remains a vertical asymptote despite the isolated point $(-1, 6)$.

Check your understanding

1.14 C 1 If a denominator vanishes at a, what extra check is needed before naming a vertical asymptote?

1.14 C 2 Suppose $g(x) = 1/\sqrt{x-2}$ on its domain. Does it have a vertical asymptote at 2 even though no values lie just to the left?

Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.15

End behavior and horizontal asymptotes

Evaluate limits at each end of the domain, identify horizontal asymptotes, and use ratios to compare growth.

LIM-2.D • OpenStax Calculus Volume 1, §4.6, Limits at Infinity and Asymptotes

In a limit at infinity, the input travels indefinitely in one direction. A finite output limit L gives the horizontal asymptote $y = L$ on that end. The two ends can have different limiting heights. For a rational function, divide numerator and denominator by the highest power in the denominator and use $1/x^k \to 0$ for positive integer k. For a radical remember $\sqrt{x^2} = |x|$, which changes how signs work at negative infinity. To compare growth of positive quantities, take their ratio: a limit of zero means the numerator becomes negligible relative to the denominator, even when both grow.

Worked example 1

Find the end behavior of $F(x) = (6x^2 - x + 1)/(2x^2 + 5)$ and identify any horizontal asymptote.

1. Divide top and bottom by x^2 to get $(6 - 1/x + 1/x^2)/(2 + 5/x^2)$.
2. At both positive and negative infinity the fractions involving x tend to 0. Both limits are $6/2 = 3$, giving $y = 3$.

Worked example 2

Find the horizontal asymptotes of $g(x) = x/\sqrt{x^2 + 4}$. Why do the two ends differ?

1. Factor x^2 in the radical for $x \neq 0$: $g(x) = (x/|x|)/\sqrt{1 + 4/x^2}$.
2. At positive infinity the sign factor is 1; at negative infinity it is -1. The respective asymptotes are $y = 1$ and $y = -1$.

Worked example 3

Two positive quantities are $A(t) = t^2$ and $B(t) = t^3$ for $t > 0$. Their rates are given by $A'(t) = 2t$ and $B'(t) = 3t^2$. Compare the quantities and, separately, their rates as $t \to \infty$.

1. The ratio of the quantities is $A(t)/B(t) = 1/t \to 0$. Thus A becomes small relative to B, although $A(t) \to \infty$ and $B(t) \to \infty$. A zero ratio does not mean the numerator approaches zero.
2. The ratio of their given rates is $A'(t)/B'(t) = 2/(3t) \to 0$. Both rates grow without bound, but the rate of A becomes small relative to the rate of B. These are two comparisons of different functions.

Check your understanding

1.15 C 1 Can a curve cross a horizontal asymptote? For $p(x) = 2 + x/(x^2 + 1)$, use algebra to establish both end limits. Then identify an input where the curve crosses its asymptote and explain why this does not contradict either limit.

1.15 C 2 For large positive t, compare the given rates $r(t) = 3t^2$ and $s(t) = 4t^3$ using a limit of r/s.
Explain your reasoning before turning to the check key.

LESSON 02 / CED 1.16

The Intermediate Value Theorem

Use endpoint information and continuity to guarantee an intermediate output, and state the strength of the conclusion.

FUN-1.A • OpenStax Calculus Volume 1, §2.4, Continuity

Suppose f is continuous on $[a, b]$, and a desired value d lies strictly between $f(a)$ and $f(b)$. The Intermediate Value Theorem guarantees at least one c in (a, b) for which $f(c) = d$. In an argument, specify the interval, the reason for continuity, the endpoint values and the value between them. The theorem gives existence. Other information is needed to locate the input exactly or prove it is unique. When the desired value equals an endpoint value, that endpoint already supplies a solution.

Worked example 1

Prove there is a solution of $x^3 + x = 9$ between 1 and 2 without finding the solution.

1. Set $f(x) = x^3 + x$, which is continuous on $[1, 2]$ because it is a polynomial. The endpoint values are 2 and 10.
2. Since $2 < 9 < 10$, the IVT provides a c in $(1, 2)$ with $f(c) = 9$.

Worked example 2

A table for a continuous temperature records 9°C at 7 a.m., 15°C at noon, and 10°C at 3 p.m. How many distinct times must it have been 12°C?

1. On the interval from 7 a.m. to noon, the endpoint values straddle 12. Continuity and the IVT give a time strictly inside this interval with temperature 12°C.
2. The endpoint values on the noon-to-3 interval also straddle 12. The two open intervals are disjoint, so at least two distinct times are guaranteed.

Check your understanding

1.16 C 1 Why do $g(-1) = -1$ and $g(1) = 1$ fail to prove a zero for $g(x) = 1/x$ on $[-1, 1]$?

1.16 C 2 A polynomial has $p(2) = 4$ and $p(6) = 11$. Use the IVT to state exactly what you can guarantee about $p(x) = 10$. Would the same endpoint data guarantee $p(x) = 15$?

Explain your reasoning before turning to the check key.

LESSON 03

From secants to derivatives

Explain the derivative as a limit of average rates and use it to estimate a rate or write a tangent line.

CED 2.1–2.4 • OpenStax Calculus Vol. 1, sections 3.1–3.2

The idea

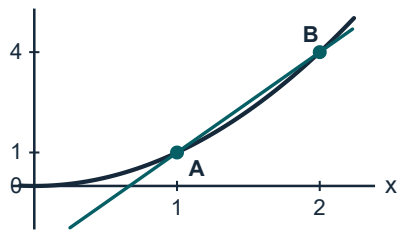
Over $[a, b]$ with $b \neq a$, the average rate is $(f(b) - f(a))/(b - a)$. It is the slope of the secant through the two graph points. If the output is in meters and the input is in seconds, the quotient is in meters per second.

At an interior point a, the derivative is $f'(a) = \lim_{h \to 0} (f(a+h) - f(a))/h$ when this limit exists as a finite number. An equivalent form uses $(f(x) - f(a))/(x - a)$ as $x \to a$. In each, the numerator is a change in output and the denominator is the matching change in input.

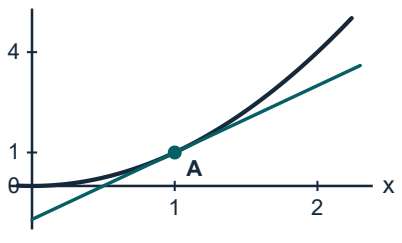
A derivative value is a slope. A tangent line also needs a point. Its equation at $x = a$ is $y = f(a) + f'(a)(x - a)$. A derivative function f' gives a slope at each input where the derivative exists. Positive slopes on an interval indicate an increasing function there.

Differentiability implies continuity. At a corner, the one-sided slopes differ. At a vertical tangent a finite slope is unavailable. A discontinuity also rules out differentiability. A horizontal tangent has finite slope zero and is compatible with differentiability.

For a table, use nearby points on opposite sides if available and compute the secant slope. It estimates the instantaneous rate. On a graph, estimate the slope of a drawn tangent using two widely separated points on that line.



Secant through A and B



Tangent at A

Both panels show $y = x^2$ with $A = (1, 1)$. The left line also passes through $B = (2, 4)$.

Find the secant slope. If B moves along the curve toward A, which value should the slope approach?

LESSON 03 / WORKED EXAMPLES

From secants to derivatives

Example 1

Find the slope and equation of the tangent to $f(x) = 1/x$ at $x = 2$ from the definition.

1. The difference quotient is $\left[1/(2+h) - 1/2\right]/h$. Combine the fractions in the numerator to get $-h/[2(2+h)]$, then divide by h.
2. For $h \neq 0$ near zero, the quotient is $-1/[2(2+h)]$. Its limit is $-1/4$.
3. Since $f(2) = 1/2$, the tangent is $y = 1/2 - (x-2)/4$.

The cancellation uses nonzero h. The limit, taken afterward, supplies the instantaneous slope.

Example 2

At 2.8, 3.0, and 3.2 seconds a cart's positions are 18.9, 20.0, and 21.5 meters. Estimate its velocity at 3.0 seconds and explain what you can claim.

1. Use the values at 2.8 and 3.2 seconds to straddle the target. The secant slope is $(21.5 - 18.9)/(3.2 - 2.8) = 6.5$ m/s.
2. This is the exact average velocity over the 0.4-second interval and an estimate of the instantaneous velocity at 3.0 seconds.

Several smooth position curves can pass through the same measurements with different instantaneous velocities.

> **Common error.** A line tangent to a curve can cross it. Tangency is determined by the slope at the point. For example, $y = 0$ is tangent to $y = x^3$ at the origin.

Explain it aloud

Use the corner in the graph of $y = |x|$ to explain why continuity alone cannot give a tangent slope.

LESSON 03 / CED 2.1

From an average rate to an instantaneous rate

Calculate a rate over an interval and build both limit forms for a rate at one instant.

CHA-2.A / CHA-2.B • OpenStax Calculus Vol. 1, §§3.1, 3.4

Read a point on a distance or temperature graph as a measurement. Read the rise divided by the run between two measurements as an average rate. For an input change $h \neq 0$ from a, the average is $[f(a+h) - f(a)]/h$. It has units of output per unit of input. A negative rate describes a decrease as the input increases.

To pinpoint the rate at a, let the second input approach a. If the slopes approach one finite value from both sides, that value is $f'(a)$. With $x = a + h$, the same limit is $\lim_{x \to a} [f(x) - f(a)]/(x - a)$. The denominator is nonzero during the limiting process.

Worked example 1

A pump test records $V(t) = 60 - 2t^2$ L for $0 \leq t \leq 4$ minutes. Compare the average rate from 1 to 3 minutes with that between times 1 and $1 + h$ for small $h \neq 0$. Find the instantaneous rate at minute 1.

1. $V(1) = 58$ and $V(3) = 42$. The average rate on $[1, 3]$ is $(42 - 58)/2 = -8$ L/min.
2. $V(1+h) - V(1) = -4h - 2h^2$. Divide by h to get $-4 - 2h$ for $h \neq 0$.
3. As $h \to 0$ this approaches -4 L/min. The water volume falls faster on average over the longer interval.

Worked example 2

The area of a square of side s is $A(s) = s^2$. Write the rate of change of area at $s = 5$ using a moving endpoint u.

1. The line through $(5, 25)$ and (u, u^2) has slope $(u^2 - 25)/(u - 5)$.
2. For $u \neq 5$ the slope equals $u + 5$. Its limit as $u \to 5$ is 10 cm² per centimeter if sides are in cm.
3. The change in side length is $u - 5$; calling it h gives the same limit.

Check your understanding

2.1 C 1 Height changes from 1.8 m at 2 seconds to 1.2 m at 5 seconds. Determine the average rate.

2.1 C 2 For $f(x) = 1/x$, write its instantaneous rate at $x = 4$ in both limit forms.

2.1 C 3 If $R(t) = 7t - t^2$, form the average-rate quotient between times 2 and $2 + h$ for $h \neq 0$. Determine its limit as h approaches zero from either side.

Explain your reasoning before turning to the check key.

LESSON 03 / CED 2.2

The derivative function and the tangent line

Construct a derivative function from its definition and use its value and a point to write a tangent equation.

CHA-2.B / CHA-2.C • OpenStax Calculus Vol. 1, §§3.1–3.2

Keep x variable in $f'(x) = \lim_{h \to 0} [f(x+h) - f(x)]/h$. The result assigns a slope to each input where the limit exists. At a particular input a, $f'(a)$ is one number. For $y = f(x)$, the symbols $f'(x)$, y' and dy/dx represent the derivative function.

A tangent line needs two pieces of information: its slope $f'(a)$ and its point $(a, f(a))$. Use $y - f(a) = f'(a)(x - a)$. A tangent may cross the curve. To sketch f', plot the slope of the original graph as a new vertical coordinate at each input.

Worked example 1

Use the definition to find the derivative function of $f(x) = x^2 + 4x$, then write its tangent line at $x = -1$.

1. Substitute $x + h$ for every x: $f(x+h) = (x+h)^2 + 4(x+h) = x^2 + 2xh + h^2 + 4x + 4h$.
2. Subtract the whole original output: $f(x+h) - f(x) = [x^2 + 2xh + h^2 + 4x + 4h] - [x^2 + 4x] = 2xh + h^2 + 4h$. The subtraction applies to every term in the second bracket.
3. Factor the numerator as $h(2x + h + 4)$. For $h \neq 0$, the difference quotient is $2x + h + 4$. Taking its limit as $h \to 0$ gives $f'(x) = 2x + 4$.
4. At $x = -1$, the slope is $f'(-1) = 2$ and the point is $(-1, -3)$. The tangent is $y + 3 = 2(x + 1)$.

Worked example 2

A curve passes through $(3, 8)$. In a small neighborhood it falls by about 0.6 unit for each 0.2-unit increase in x. Use this information to model the tangent.

1. The approximate tangent slope is $-0.6/0.2 = -3$. It is a derivative estimate because the changes are approximate.
2. The estimated line through the given point is $y - 8 = -3(x - 3)$.
3. Its prediction at $x = 3.1$ is 7.7, consistent with a locally decreasing curve.

Worked example 3

Identify the function and point represented by $\lim_{h \to 0} [\sqrt{16 + h} - 4]/h$, then construct the tangent line at that point.

1. The numerator compares $f(16 + h)$ with $f(16)$ for $f(x) = \sqrt{x}$. The point on the curve is $(16, 4)$.
2. Rationalizing gives $1/(\sqrt{16 + h} + 4)$, whose limit is $1/8$.
3. The tangent is $y - 4 = (x - 16)/8$.

Check your understanding

2.2 C 1 If $f'(x) = 6x - 1$ and $f(2) = 4$, what is the tangent line at 2?

LESSON 03 / CED 2.2 / CONTINUED

The derivative function and the tangent line

Construct a derivative function from its definition and use its value and a point to write a tangent equation.

CHA-2.B / CHA-2.C • OpenStax Calculus Vol. 1, §§3.1–3.2

2.2 C 2 On an interval a graph is a line of slope -2. What is the graph of the derivative on the interior of that interval?

2.2 C 3 Write $f'(x)$ as a limit for $f(x) = 1 - 2x$, then simplify. How does the result describe the graph?

2.2 C 4 For a differentiable function with $f(5) = 7$, suppose $\lim_{h \to 0} [f(5+h) - 7]/h = -3$. Identify the derivative in notation and write the tangent equation.

Explain your reasoning before turning to the check key.

LESSON 03 / CED 2.3

Estimating a derivative from measurements or a graph

Estimate tangent slopes from data, a plotted tangent, and a numerical derivative tool.

CHA-2.D • OpenStax Calculus Vol. 1, §§3.1–3.2, 3.4

For a table, compare output values immediately to either side of the target when possible. Use the full change in input as the denominator. A secant through two neighboring measurements estimates the local slope; the table alone cannot prove the exact derivative. If the samples are uneven, identify their actual distances from the target before judging the estimate.

On a graph, approximate the tangent line at the target, then choose two well-separated readable points on that line to estimate its slope. For a formula, a graphing calculator's numerical derivative or a centered quotient $[f(a+h) - f(a-h)]/(2h)$ supplies a numerical estimate. Compare several moderate step sizes and retain units.

Worked example 1

A temperature sensor reads 19.40°C at 3.8 min, 19.55°C at 4.0 min, and 19.74°C at 4.2 min. Estimate the instantaneous warming rate at 4.0 min.

1. The left rate is $(19.55 - 19.40)/0.2 = 0.75$ and the right rate is $(19.74 - 19.55)/0.2 = 0.95$ °C/min.
2. A centered estimate uses both sides: $(19.74 - 19.40)/0.4 = 0.85$ °C/min.
3. Since the measurements are discrete, report this value as an estimate.

Worked example 2

Estimate the derivative of $q(x) = \ln\left(1 + x^2\right)$ at $x = 1$ using a calculator and $h = 0.01$.

1. Enter $\left[\ln\left(1 + 1.01^2\right) - \ln\left(1 + 0.99^2\right)\right]/0.02$ with parentheses around the difference.
2. The result is about 0.999983. Try $h = 0.001$ to obtain about 0.9999998.
3. Both support a local slope of approximately 1.

Worked example 3

For $f(x) = x^2$, compare two secant estimates of the tangent slope at $x = 3$. Why does their average happen to be exact?

1. Use endpoints 2.9 and 3.1 separately with 3. Since $\left(x^2 - 9\right)/(x - 3) = x + 3$ when $x \neq 3$, the slopes are 5.9 and 6.1.
2. The secant slopes approach 6 as the second point approaches 3. Their average is also 6 because $x + 3$ is linear and the endpoints lie equally far from 3.
3. For a general curved function, a centered estimate can still differ from the true derivative.

Adapted from OpenStax, Calculus Volume 1, §3.1, Example 3.4. Added the centered-estimate argument.
Adapted material: CC BY-NC-SA 4.0.

Access for free at https://openstax.org/books/calculus-volume-1/pages/3-1-defining-the-derivative

LESSON 03 / CED 2.3 / CONTINUED

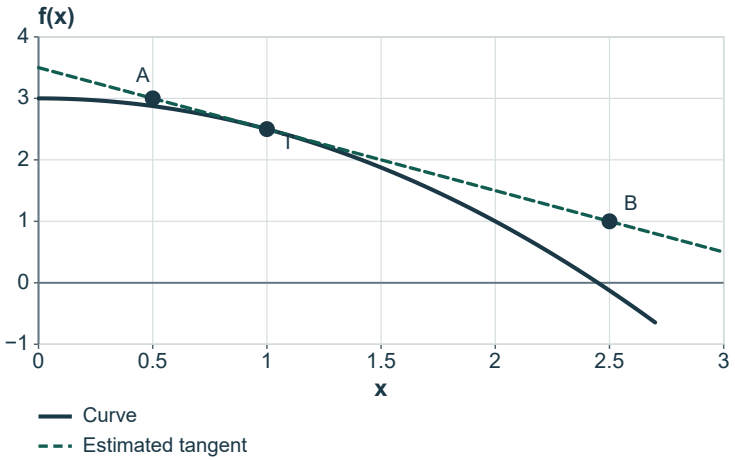
Estimating a derivative from measurements or a graph

Estimate tangent slopes from data, a plotted tangent, and a numerical derivative tool.

CHA-2.D • OpenStax Calculus Vol. 1, §§3.1–3.2, 3.4

Worked example 4

Estimate the derivative at the marked point $T = (1, 2.5)$ on the smooth curve. The dashed line shows a tangent fitted at T. Explain which points you use for the slope calculation.



1. Place the tangent through T so that its direction agrees with the curve close to T. The dashed line is a local model; it need not follow the curve farther away.
2. Choose readable points on the tangent, not on the curved graph. Here $A = (0.5, 3)$ and $B = (2.5, 1)$ lie on the dashed line. They are far enough apart that small reading errors have less effect.
3. The estimated slope is $(1 - 3) / (2.5 - 0.5) = -1$. Therefore $f'(1) \approx -1$. The curve falls by about one output unit per input unit near $x = 1$.
4. The tangent and curve share T, but A and B were selected on the tangent. Taking two other points on the curve would estimate a secant slope instead.

Check your understanding

2.3 C 1 A tangent line drawn on a graph passes close to $(2, 7)$ and $(5, 1)$. Estimate the derivative where it touches the curve.

2.3 C 2 A sensor gives $f(8.8) = 10.2$ and $f(9.4) = 11.4$. Estimate $f'(9)$ from these data. Explain why the estimate need not equal the exact derivative.

Explain your reasoning before turning to the check key.

LESSON 03 / CED 2.4

Where differentiability fails

Decide whether a derivative exists and justify the decision with continuity or one-sided slopes.

FUN-2.A • OpenStax Calculus Vol. 1, §3.2

Differentiability at an interior point requires a finite two-sided limit of the difference quotient. If f has such a limit at a, then $f(a+h) - f(a) = h \cdot [f(a+h) - f(a)]/h$ tends to zero as $h \to 0$. Thus f is continuous at a. A hole or jump rules out differentiability.

After checking continuity, test the slopes. A corner has unequal finite one-sided slopes. At a cusp or a vertical tangent, at least one side is unbounded. In each case the usual finite derivative is absent. A horizontal tangent has finite slope zero.

Worked example 1

Let $p(x) = x^2$ for $x \leq 1$ and $p(x) = mx + b$ for $x > 1$. Choose m, b to make p differentiable at 1.

1. Continuity requires $m + b = 1$. Matching function values alone leaves many possibilities.
2. The left slope is 2, and the right slope is m. Matching slopes gives $m = 2$.
3. Hence $b = -1$. Both sides meet with the same slope, so $p'(1) = 2$.

Worked example 2

Does $f(x) = \sqrt[3]{x}$ have a derivative at zero? Account for continuity and the difference quotient.

1. It is continuous at zero. For $h \neq 0$, the quotient from the origin is $\sqrt[3]{h}/h = 1/\left(\sqrt[3]{h}\right)^2$.
2. The quotient increases without bound from both sides. There is a vertical tangent but no finite $f'(0)$.

Check your understanding

2.4 C 1 A curve has a removable hole at $x = 3$. Could it be differentiable there before filling the hole?

2.4 C 2 If both one-sided difference quotients at a tend to zero, can there be a corner there?
Explain your reasoning before turning to the check key.

LESSON 04

Choosing a differentiation rule

Identify the outermost operation, select a derivative rule, and check the result against the structure of the function.

CED 2.5–2.10 • OpenStax Calculus Vol. 1, sections 3.3, 3.5, 3.9

The idea

Constants differentiate to zero. Differentiate sums term by term and keep constant multiples. The power rule gives $(x^n)' = nx^{n-1}$ wherever this derivative formula is defined. A negative exponent can simplify a reciprocal before differentiation, and a square root is a power of $1/2$.

For differentiable functions u and v, $(uv)' = u'v + uv'$. When $v \neq 0$, $\left(u/v\right)' = (u'v - uv')/v^2$. With a polynomial over a single power of x, dividing term by term first often gives a shorter route. Preserve the original domain.

At inputs where the formulas are defined, the basic exponential and log derivatives include $(e^x)' = e^x$, $(a^x)' = a^x \ln a$ for $a > 0$, and $(\ln x)' = 1/x$ for $x > 0$. For base $a > 0$ with $a \neq 1$, $\left(\log_a x\right)' = 1/(x \ln a)$.

In radians, the derivatives of sine, cosine, and tangent are $\cos x$, $-\sin x$, and $\sec^2 x$. Those of cotangent, secant, and cosecant are $-\csc^2 x$, $\sec x \tan x$, and $-\csc x \cot x$, respectively. Match the name and the sign before substituting a special angle.

The second derivative f'' measures how the first derivative changes. Find an expression for f' and differentiate it again. A table of values for f, g, f', and g' at one input lets you evaluate product or quotient derivatives without knowing either original formula.

> **Think it through.** Explain why you can differentiate $\left(x^3 + x\right)/x$ by two routes and why neither route gives a derivative of the original function at zero.

LESSON 04 / WORKED EXAMPLES

Choosing a differentiation rule

Example 1

Differentiate $F(x) = x^2 \cos x$ and find $F'(\pi/2)$.

1. The outermost operation is multiplication. The product rule gives $F'(x) = 2x \cos x - x^2 \sin x$.
2. Use $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. The result is $F'(\pi/2) = -\pi^2/4$.

Each term of the product rule describes the effect of changing one factor while holding the other at its current value.

Example 2

At $x = 1$, suppose $u = 6$, $v = 2$, $u' = -1$, and $v' = 3$. Find the derivative of u/v at 1.

1. The denominator is nonzero. Keep the order $u'v - uv'$ in the numerator.
2. Substitute the values to get $[(-1)(2) - 6(3)]/2^2 = -5$.

The numerator decreases while the positive denominator increases, so the negative answer has the expected sign.

> **Common error.** The product of two derivatives does not give the derivative of the product. For $x \cdot x$ that shortcut gives 1, but differentiating x^2 gives $2x$.

LESSON 04 / CED 2.5

The power rule

Differentiate a power and check where both the original function and its derivative make sense.

FUN-3.A • OpenStax Calculus Vol. 1, §§3.3, 3.7

For a fixed real exponent r, the derivative of x^r is rx^{r-1} wherever the rule applies; positive x always works. A power of x has a variable base and a constant exponent. Rewrite radicals and reciprocals using exponents before differentiating. For example $1/x^3 = x^{-3}$.

Check domain restrictions in the original expression and the derivative. A root may permit $x = 0$ in the function while the new expression has no finite value there. At a questionable point use the definition. For a positive integer power the rule also follows from expanding $(x+h)^r - x^r$, dividing by h, and taking the limit.

Worked example 1

Find the slope of $y = x^{5/3}$ at $x = -8$. Is a negative input allowed?

1. The cube root of a negative number exists. Apply the power rule to get $y' = (5/3)x^{2/3}$.
2. Since $\sqrt[3]{-8} = -2$, the slope is $(5/3)(-2)^2 = 20/3$.

Worked example 2

For $R(x) = 1/\sqrt{x}$, find the derivative at 4 and explain its sign.

1. For $x > 0$ write $R(x) = x^{-1/2}$. Then $R'(x) = -(1/2)x^{-3/2}$.
2. $R'(4) = -1/16$. As x grows, its positive square root grows while its reciprocal falls.

Check your understanding

2.5 C 1 Find $d(x^{-5})/dx$, stating the allowed inputs.

2.5 C 2 Someone writes $d\left(x^{7/2}\right)/dx = (7/2)x^{7/3}$. Which step failed?

Explain your reasoning before turning to the check key.

LESSON 04 / CED 2.6

Differentiating sums and constant multiples

Use linearity and the power rule to differentiate combinations of functions, including those given by data.

FUN-3.A • OpenStax Calculus Vol. 1, §3.3

A constant has derivative zero because its output change is zero. A fixed multiplier scales every output change. Adding functions adds their changes. Therefore $(af + bg + c)' = af' + bg'$ when a, b, c are constants and the component derivatives exist. A difference is a sum with a negative coefficient.

Mark each term, including its sign, before taking derivatives. A constant term disappears; a linear term contributes its coefficient. When coefficients are parameters, hold them fixed while differentiating with respect to the stated variable.

Worked example 1

Find $p'(x)$ and $p'(-2)$ if $p(x) = -2x^4 + 5x^2 - 7x + 12$.

1. Differentiate by terms to obtain $p'(x) = -8x^3 + 10x - 7$.
2. Substitution gives $p'(-2) = 64 - 20 - 7 = 37$. The constant 12 does not affect the slope.

Worked example 2

The weight of a package is modeled by $M(t) = 3A(t) - 2B(t) + 40$. If $A'(5) = 1.2$ and $B'(5) = -0.4$ in g/min, find $M'(5)$.

1. The constant packaging contribution has zero rate. Thus $M' = 3A' - 2B'$.
2. $M'(5) = 3(1.2) - 2(-0.4) = 4.4$ g/min. A decreasing B term increases this difference.

Check your understanding

2.6 C 1 Differentiate $y = ax^3 - bx + c$ with respect to x when a, b, c are constants.

2.6 C 2 Two curves differ by a constant vertical shift on an interval. What can you conclude about their derivatives wherever these exist?

Explain your reasoning before turning to the check key.

LESSON 04 / CED 2.7

Trig, exponential, and logarithmic derivatives

Differentiate elementary functions and recognize their derivatives inside limits.

FUN-3.A / LIM-3.A • OpenStax Calculus Vol. 1, §§3.1, 3.5, 3.9

With angles in radians, $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. The other basic pairs are $(e^x)' = e^x$ and $(\ln x)' = 1/x$ for $x > 0$. For a positive base $b \neq 1$, $b^x = e^{x \ln b}$ gives $\left(b^x\right)' = b^x \ln b$ and $\left(\log_b x\right)' = 1/(x \ln b)$.

Look for differences of the same function at nearby inputs. If the numerator is $f(a+h) - f(a)$ and the denominator is h, the limit is $f'(a)$. Confirm the variable increment, the base point, and the order of the subtraction. A scaled input increment can leave a constant factor.

Worked example 1

For $F(x) = 4\sin x - \cos x + e^x - 2\ln x$, find $F'(x)$ and its domain.

1. Differentiate each term: $4\cos x + \sin x + e^x - 2/x$.
2. All terms must be defined in the original function, so use $x > 0$.

Worked example 2

Evaluate $\lim_{h \to 0} \left[\cos\left(\pi/6 + 2h\right) - \cos\left(\pi/6\right)\right]/h$.

1. Set $u = 2h$. Replacing h in the denominator gives twice the quotient with denominator u.
2. The remaining limit is the derivative of cosine at $\pi/6$, namely $-\sin\left(\pi/6\right) = -1/2$.
3. Multiplying by 2 gives -1.

Worked example 3

Find $g'(x)$ for $g(x) = \sin x + 2^x$, then evaluate $\lim_{h \to 0} \left[g(\pi + h) - g(\pi)\right]/h$.

1. By the sum rule, $g'(x) = \cos x + 2^x \ln 2$.
2. The quotient is the definition of $g'(\pi)$; the increment in the numerator and denominator is the same.
3. Substituting gives $-1 + 2^\pi \ln 2$.

Check your understanding

2.7 C 1 Find the derivative of $3^x + \log_2 x$.

2.7 C 2 Interpret $\lim_{x \to 2} (\ln x - \ln 2)/(x-2)$, then evaluate it.

2.7 C 3 If $F(x) = \cos x + \ln x$, what value does $\lim_{h \to 0} [F(1+h) - F(1)]/h$ approach? Write the derivative used.

Explain your reasoning before turning to the check key.

LESSON 04 / CED 2.8

The product rule

Account for both contributions to a product's instantaneous change.

FUN-3.B • OpenStax Calculus Vol. 1, §3.3

For differentiable u and v, $(uv)' = u'v + uv'$. Each term describes how changing one factor affects the product at the old value of the other. In a finite increment, the product change includes an additional $\Delta u \Delta v$. Dividing by the input increment and taking the limit makes this additional contribution vanish.

For functions supplied by a table, gather both values and both derivatives at the same input. Multiplying only the two derivatives loses the two first-order contributions. When a product is easy to expand, differentiating the expansion provides an independent check.

Worked example 1

Find the slope of $y = \left(x^2 - 1\right)e^x$ at $x = 1$.

1. Apply the rule to obtain $y' = 2xe^x + \left(x^2 - 1\right)e^x$.
2. At 1, $y' = 2e + 0 = 2e$. The product has value zero there, yet changing its first factor still affects its slope.

Worked example 2

A rectangle is 5 m wide and 8 m long. Width grows at 0.3 m/s while length shrinks at 0.2 m/s. Find the area rate.

1. Write $A = WL$. Its rate is $A' = W'L + WL'$.
2. $A' = 0.3\,(8) + 5\,(-0.2) = 1.4$ m²/s. The gain from width exceeds the loss from length.

Check your understanding

2.8 C 1 At 4, $u = 0$, $v = -3$, $u' = 2$, $v' = 10$. Find $(uv)'(4)$.

2.8 C 2 Differentiate $x \sin x$ and evaluate the derivative at π.
Explain your reasoning before turning to the check key.

LESSON 04 / CED 2.9

The quotient rule

Differentiate a quotient, preserving subtraction order and domain restrictions.

FUN-3.B • OpenStax Calculus Vol. 1, §3.3

For $y = u/v$ with $v \neq 0$, rewrite the relationship as $yv = u$. Differentiating gives $y'v + yv' = u'$. Solving for y' produces $(u'v - uv')/v^2$. This derivation explains both the order of subtraction and the squared denominator.

Keep the numerator of the derivative in parentheses until after substitution. Algebraic cancellation before differentiating can save work, but it cannot restore any input excluded from the original function.

Worked example 1

Find $F'(x)$ for $F(x) = (x^2 + 1)/(x - 2)$ and determine $F'(0)$.

1. Set $u = x^2 + 1$ and $v = x - 2$. Then $u' = 2x$ and $v' = 1$.
2. $F' = [2x(x - 2) - (x^2 + 1)]/(x - 2)^2 = (x^2 - 4x - 1)/(x - 2)^2$ for $x \neq 2$.
3. At zero the result is $-1/4$.

Worked example 2

A container holds 30 g of salt in 6 L of water. At that instant the salt and volume rise at 2 g/min and 0.5 L/min. Find the rate of change in salt concentration.

1. Concentration is $C = S/V$. Use $C' = (S'V - SV')/V^2$.
2. $C' = [2(6) - 30(0.5)]/36 = -1/12$ g per L per minute. The concentration falls even though the mass of salt increases.

Check your understanding

2.9 C 1 Where does the derivative of $(x^2 - 4)/(x - 2)$ equal 1?

2.9 C 2 Suppose $p(0) = 4$, $q(0) = 2$, $p'(0) = -1$, $q'(0) = 3$. Evaluate $(p/q)'(0)$.
Explain your reasoning before turning to the check key.

LESSON 04 / CED 2.10

Derivatives of the four remaining trig functions

Recover the four derivative rules from sine, cosine and the quotient rule, then evaluate them with radian inputs.

FUN-3.B • OpenStax Calculus Vol. 1, §3.5

Write $\tan x = \sin x / \cos x$ and $\cot x = \cos x / \sin x$. The quotient rule and $\sin^2 x + \cos^2 x = 1$ yield $(\tan x)' = \sec^2 x$ and $(\cot x)' = -\csc^2 x$. Differentiating $1/\cos x$ and $1/\sin x$ gives $(\sec x)' = \sec x \tan x$ and $(\csc x)' = -\csc x \cot x$.

The denominator in each starting identity tells you where its rule applies. Tangent and secant require $\cos x \neq 0$; cotangent and cosecant require $\sin x \neq 0$. Write angles in radians when applying these rules.

Worked example 1

Derive the cotangent derivative from its quotient and estimate its sign without a calculator.

1. For $\sin x \neq 0$, write $\cot x = \cos x / \sin x$.
2. The derivative is $\left[-\sin^2 x - \cos^2 x\right] / \sin^2 x = -1/\sin^2 x$.
3. Thus $(\cot x)' = -\csc^2 x$, strictly negative wherever cotangent is defined.

Worked example 2

Derive the secant rule and use it at $x = 2\pi/3$.

1. On $\cos x \neq 0$, differentiate $1/\cos x$ to get $\sin x / \cos^2 x = \sec x \tan x$.
2. At $2\pi/3$, $\sec x = -2$ and $\tan x = -\sqrt{3}$. The derivative is $2\sqrt{3}$.

Check your understanding

2.10 C 1 What is the derivative of $\csc x$ at $\pi/6$?

2.10 C 2 Evaluate $d(\tan x)/dx$ at $x = 3\pi/4$. Why is the answer positive there?

Explain your reasoning before turning to the check key.

LESSON 05

Chain, implicit, and inverse rules

Track the input at each layer, distinguish an inverse from a reciprocal, and find slopes when both coordinates appear in an equation.

CED 3.1–3.6 • OpenStax Calculus Vol. 1, sections 3.2, 3.6–3.8

The idea

For a composition $y = f\big(g(x)\big)$, label the inner output $u = g(x)$. A small change in x changes u at rate $g'(x)$, and u changes the output at rate $f'(u)$. Multiplying gives $y' = f'\big(g(x)\big)g'(x)$. Each extra layer contributes another factor.

In an equation containing x and y, treat y as a differentiable function of x near the point of interest. Differentiate both sides with respect to x, apply the chain rule to every term containing y, and collect the terms with y'. If you divide by an expression, check it at the point.

If $g = f^{-1}$ near a point, then $g(f(a)) = a$. Under the inverse derivative rule, when f has a differentiable local inverse and $f'(a) \neq 0$, $g'(f(a)) = 1/f'(a)$. Match a target input for g to an output of f. The graphs have reciprocal slopes at reflected points.

Principal inverse trig functions return angles in radians. For differentiable $u = u(x)$ and $|u| < 1$, $(\arcsin u)' = u'/\sqrt{1-u^2}$ and $(\arccos u)' = -u'/\sqrt{1-u^2}$. For any real u, $(\arctan u)' = u'/\big(1+u^2\big)$. The inverse of sine returns an angle, while the reciprocal of sine is cosecant.

> **Think it through.** Compare the three operations in $\sin\left(x^2\right)$, $(\sin x)^2$, and $\arcsin x$. Name the outer operation before you describe any derivative.

LESSON 05 / WORKED EXAMPLES

Chain, implicit, and inverse rules

Example 1

A differentiable function f satisfies $f'(4) = -3$. Find the derivative of $H(x) = f\left(x^2 + 3\right)$ at $x = 1$.

1. The inner value at 1 is $1^2 + 3 = 4$. The inner derivative at 1 is $2(1) = 2$.
2. By the chain rule $H'(1) = f'(4) \cdot 2 = -6$.

The derivative of the outside function is evaluated at the inside output. A lookup of $f'(1)$ would use the wrong input.

Example 2

The ellipse $x^2 + 4y^2 = 20$ passes through $(2, 2)$. Find its tangent line at that point.

1. Differentiate both sides to obtain $2x + 8yy' = 0$. The factor y' comes from differentiating y^2 with respect to x.
2. Since $y = 2 \neq 0$, $y' = -x/\left(4y\right) = -1/4$ at the point. The tangent is $y - 2 = -(x - 2)/4$.

The ellipse has two branches at $x = 2$. Specifying the point identifies the one whose slope we need.

> **Common error.** If f has a local differentiable inverse and $f'(a) \neq 0$ at $a = f^{-1}(b)$, then $\left(f^{-1}\right)'(b) = 1/f'(a)$. Check the matching input and the nonzero derivative. For example, $f(x) = x^3$ is invertible but its inverse has no finite derivative at 0.

LESSON 05 / CED 3.1

Rates through composed functions

Differentiate nested functions and evaluate each derivative at the correct intermediate input.

FUN-3.C • OpenStax Calculus Vol. 1, §3.6

A composite $f\left(g\left(x\right)\right)$ changes through two steps. A small change in x produces a change in g, and the outer function converts that change into a change in output. At inputs where the component derivatives exist, the resulting rate is $f'\left(g\left(x\right)\right)g'\left(x\right)$. Evaluate the outer derivative at $g\left(x\right)$.

To trace several layers, write a short chain such as $x \to u \to v \to y$. Differentiate each link and multiply the factors. When u is a function of time, the intermediate units cancel in $\left(dy/du\right)\left(du/dt\right)$, leaving output per unit time.

Worked example 1

Differentiate $y = e^{\sin\left(3x\right)}$ and find its slope at zero.

1. Trace $x \to 3x \to \sin\left(3x\right) \to e^{\sin\left(3x\right)}$.
2. Multiplying derivatives gives $y' = e^{\sin\left(3x\right)}\cos\left(3x\right)\cdot 3$.
3. At zero, the exponential and cosine both equal 1, so the slope is 3.

Worked example 2

A conversion $y = F\left(u\right)$ uses a sensor output $u = G\left(t\right)$. At $t = 4$, $G\left(4\right) = 9$, $G'\left(4\right) = -2$. The calibration table gives $F'\left(4\right) = 8$, $F'\left(9\right) = 3$. Find dy/dt.

1. Differentiating the composition gives $F'\left(G\left(4\right)\right)G'\left(4\right)$.
2. The outer input is 9, so the rate is $3\left(-2\right) = -6$ output units per unit time.

Check your understanding

3.1 C 1 Differentiate $\cos\left(x^4\right)$, and name the factor from the inside.

3.1 C 2 For differentiable f, g, $g\left(2\right) = 5$, $g'\left(2\right) = 0$, $f'\left(5\right) = 12$. What is $\left(f \circ g\right)'\left(2\right)$?

Explain your reasoning before turning to the check key.

LESSON 05 / CED 3.2

Differentiating an implicit relation

Treat y as a local function of x, differentiate the relation, then extract the slope.

FUN-3.D • OpenStax Calculus Vol. 1, §3.8

An equation can define a curve without isolating y. Near a point where it gives a differentiable branch $y(x)$, differentiate both sides with respect to x. A term that contains y needs the chain rule: $(y^3)' = 3y^2y'$. A mixed term such as xy needs the product rule as well.

Gather the terms containing y', factor, and divide only when the coefficient is nonzero. Check that a proposed point satisfies the original equation before using it to find a tangent line. A vanished coefficient calls for inspecting the original curve.

Worked example 1

Find the slope of $x^2 + 2xy + 3y^2 = 6$ at $(1, 1)$.

1. First check $1 + 2 + 3 = 6$. Differentiation gives $2x + 2y + 2xy' + 6yy' = 0$.
2. Collect terms to obtain $(2x + 6y)y' = -2x - 2y$.
3. At $(1, 1)$ the slope is $-4/8 = -1/2$.

Worked example 2

At $(0, 0)$ the relation $e^y + x^2 = 1 + 3x$ defines a local branch. Find its tangent line.

1. Differentiate the relation: $e^y y' + 2x = 3$.
2. At the origin the exponential equals 1, so $y' = 3$.
3. The tangent through the origin is $y = 3x$.

Check your understanding

3.2 C 1 For $x^2y^3 = 4$, verify that $(2, 1)$ lies on the curve. Differentiate implicitly and find the tangent slope there.

3.2 C 2 Find the slope of $xy + y^2 = 6$ at $(1, 2)$. Check the point and show how you isolate y'.
Explain your reasoning before turning to the check key.

LESSON 05 / CED 3.3

Slope of an inverse function

Identify the matching point on the original curve and use its slope to find the inverse slope.

FUN-3.E • OpenStax Calculus Vol. 1, §3.7

If $f(a) = b$, the inverse function carries b back to a. Its graph interchanges the coordinates, so the slope at (b, a) is the reciprocal of the original slope at (a, b) when the original slope is nonzero. One-to-one behavior on the chosen interval matters; a curve with several preimages needs a specified branch.

For a differentiable inverse $g = f^{-1}$, $f\big(g(x)\big) = x$. The chain rule gives $f'\big(g(x)\big)\,g'(x) = 1$, hence $g'(b) = 1/f'(a)$. If the original derivative at a vanishes, the reciprocal formula gives no finite slope. For instance the inverse of x^3 has a vertical tangent at zero.

Worked example 1

Let $f(x) = x^3 + 2x + 4$. Without solving a cubic, find $\left(f^{-1}\right)'(7)$.

1. Since $f' = 3x^2 + 2 > 0$, f is strictly increasing. The equation $f(a) = 7$ is satisfied by $a = 1$.
2. $f'(1) = 5$. The inverse slope at 7 is therefore $1/5$.

Worked example 2

A differentiable, invertible calibration curve converts input x millimeters to output v volts. At 8 mm, $v = 3$ and $dv/dx = -0.25$ V/mm. Estimate the inverse's local response to a 0.04 V increase.

1. At 3 V the inverse has slope $dx/dv = 1/(-0.25) = -4$ mm/V.
2. A local linear estimate gives $\Delta x \approx (-4)(0.04) = -0.16$ mm. Interchanging input and output also interchanges their units.

Check your understanding

3.3 C 1 If $f(3) = 11$, $f'(3) = -5$, $f'(11) = 4$, find $\left(f^{-1}\right)'(11)$ assuming f has a differentiable inverse.

3.3 C 2 Why must a branch be specified before applying the inverse derivative rule to $f(x) = x^2$ at output 4?

Explain your reasoning before turning to the check key.

LESSON 05 / CED 3.4

Differentiating inverse trigonometric functions

Derive inverse-trig rules from their principal branches and differentiate compositions with valid domains.

FUN-3.E • OpenStax Calculus Vol. 1, §3.7

The usual principal branches give $(\arcsin x)' = 1/\sqrt{1-x^2}$ and $(\arccos x)' = -1/\sqrt{1-x^2}$ on $-1 < x < 1$. For all real x, $(\arctan x)' = 1/(1+x^2)$. These derivatives refer to output angles in radians. With a differentiable inner expression $u(x)$, multiply each formula evaluated at u by u'.

For completeness, on the standard principal branches $(\operatorname{arccot} x)' = -1/(1+x^2)$, $(\operatorname{arcsec} x)' = 1/\left(|x|\sqrt{x^2-1}\right)$, and $(\operatorname{arccsc} x)' = -1/\left(|x|\sqrt{x^2-1}\right)$ on $|x| > 1$ for the last two. The inverse of a trigonometric function returns an angle; a reciprocal such as $1/\sin x$ returns a ratio.

Worked example 1

Derive the arcsine derivative from $\sin y = x$. Why does the denominator use the positive square root?

1. For $-1 < x < 1$, the principal angle lies between $-\pi/2$ and $\pi/2$. Differentiation gives $\cos y\, y' = 1$.
2. Cosine is positive on this interval, so $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$.
3. Thus $y' = 1/\sqrt{1-x^2}$. The endpoints give zero cosine and no finite slope.

Worked example 2

Find $d[\arctan(3/x)]/dx$ for $x \neq 0$.

1. Set $u = 3/x$, so $u' = -3/x^2$.
2. Applying $u'/(1+u^2)$ gives $(-3/x^2)/(1+9/x^2) = -3/(x^2+9)$.
3. Retain $x \neq 0$ because the original expression includes $3/x$.

Check your understanding

3.4 C 1 At $x = 0$, what is the derivative of $\arccos(2x)$?

3.4 C 2 How do the slopes of $\arcsin x$ and $\arccos x$ compare at $x = 1/2$, and why?
Explain your reasoning before turning to the check key.

LESSON 05 / CED 3.5

Choosing and combining derivative rules

Practice CED skill 1.C by choosing an efficient first rule and explaining the order of the remaining steps.

Skill 1.C • OpenStax Calculus Vol. 1, §§3.3, 3.5–3.9

Read the outermost operation before differentiating. Ask whether the whole expression is a sum, product, quotient, composition, or relation between variables. In $x^2 \sin(3x)$, multiplication joins the main pieces, while the sine factor itself contains a composition. Handle the main pieces before their internal layers.

Try a short rewrite when it reveals a simpler structure. Preserve the original domain. A table may determine a derivative at a point even when no formula is given, provided it contains the values and derivatives needed by your chosen rule.

Worked example 1

Choose a way to differentiate $k(x) = (x^3 - 8)/(x - 2)$ on its original domain.

1. The numerator factors as $(x - 2)(x^2 + 2x + 4)$. Cancellation reduces the quotient to a polynomial for $x \neq 2$.
2. Thus $k'(x) = 2x + 2$ on the same domain. The original function has a hole at 2.

Worked example 2

Plan the order of rules for $z = x^2 \sin(x^3)$. Then carry it out.

1. The outermost operation is multiplication: differentiate each factor in turn.
2. The derivative of $\sin(x^3)$ uses the chain rule and is $3x^2 \cos(x^3)$.
3. Thus $z' = 2x \sin(x^3) + 3x^4 \cos(x^3)$.

Worked example 3

Choose a method for each task before calculating. (a) Find the slope of $x^2 + xy = 3$ at $(1, 2)$. (b) An invertible differentiable p has $p(2) = 7$ and $p'(2) = -4$. Find $(p^{-1})'(7)$.

1. (a) The equation relates x and y without isolating y, so use implicit differentiation. The point satisfies the relation. Then $2x + y + xy' = 0$, giving $y' = -4$ at $(1, 2)$.
2. (b) The input 7 belongs to the inverse. Locate its preimage using $p(2) = 7$, then take the reciprocal of the nonzero derivative: $(p^{-1})'(7) = 1/p'(2) = -1/4$.

Check your understanding

3.5 C 1 Choose the first rule for $(x + \cos x)^4$ and identify the derivative of the inner expression.

3.5 C 2 Would expanding or using a quotient rule be simpler for $(x^3 - 2x^2 + x)/x$ on $x \neq 0$? Explain.
Explain your reasoning before turning to the check key.

LESSON 05 / CED 3.6

Second and higher derivatives

Differentiate a derivative, translate higher-order notation, and interpret a second derivative from data.

FUN-3.F • OpenStax Calculus Vol. 1, §§3.2, 3.4, 3.8

At points where f' is differentiable, its derivative is f''. Equivalent notations include y'', d^2y/dx^2, and $f^{(2)}(x)$. Repeating the process produces f''' and later derivatives where they exist. An exponent written in parentheses in $f^{(4)}$ denotes derivative order.

If outputs are meters and inputs seconds, f' is measured in m/s and f'' in m/s^2. If f'' is positive throughout an interval, f' increases there; the original function might still be falling. From a graph of f' estimate f'' as its slope. If v is velocity, speed is its magnitude $|v|$. When velocity is negative and increasing toward zero, speed is decreasing; for example, moving from -9 to -8 m/s means slowing down.

Worked example 1

Find $f''(x)$ if $f(x) = xe^{2x}$. Use parentheses to keep the product and chain rules visible.

1. $f' = e^{2x} + 2xe^{2x} = (1 + 2x)\,e^{2x}$.
2. Differentiate the product again: $f'' = 2e^{2x} + 2(1 + 2x)\,e^{2x} = (4 + 4x)\,e^{2x}$.

Worked example 2

The relation $x^2 + y^2 = 10$ contains $(1, 3)$. Find d^2y/dx^2 at that point without first solving for y.

1. Differentiate and divide by 2 to obtain $x + yy' = 0$, so $y' = -1/3$ at the given point.
2. Differentiate the entire first-derivative relation: $1 + (y')^2 + yy'' = 0$.
3. $y'' = -\left(1 + 1/9\right)/3 = -10/27$.

Check your understanding

3.6 C 1 For $g(x) = \sin x$, what is $g^{(4)}(x)$?

3.6 C 2 A particle has position $s(t) = t^3 - 6t^2$ meters, with t in seconds. Find its velocity and acceleration at $t = 3$. Explain what the signs say about its motion and speed at that instant.

Explain your reasoning before turning to the check key.

LESSON 06

Rates, local estimates, and limits

Translate an instantaneous rate into a statement with units, use it for a local estimate, and recognize when a limit calls for L'Hôpital's rule.

CED 4.1–4.7 • OpenStax Calculus Vol. 1, sections 3.4, 4.1–4.2, 4.8

The idea

For one-dimensional motion, position s, velocity $v = s'$, and acceleration $a = v'$ are functions of time. Velocity carries direction and speed is $|v|$. When v and a are both nonzero, equal signs mean increasing speed and opposite signs mean decreasing speed. At a zero, inspect the surrounding behavior.

In a related-rates problem, draw and label a snapshot, then write an equation that holds as time changes. Differentiate with respect to time before inserting the snapshot's numerical measurements. State which rates are given, which rate you seek, and the units and sign of the result.

The tangent-line approximation is $f(a+h) \approx f(a) + f'(a)h$. Choose a known input a near the target. The predicted change $f'(a)h$ has output units. When the curve is concave up throughout the neighborhood, it lies on or above its tangent line. Concave down reverses the comparison.

L'Hôpital's rule applies to quotients of differentiable functions near a limiting point when the form is $0/0$ or ∞/∞ (including signed infinities), the denominator derivative is nonzero nearby, and the ratio of derivatives has a finite or infinite limit. Then the original ratio has that limit. Differentiate the numerator and denominator separately. Recheck the form before using the rule again.

> **Think it through.** A runner slows down and then reverses direction. Explain what the signs of velocity and acceleration can tell you on either side of the turn.

LESSON 06 / WORKED EXAMPLES

Rates, local estimates, and limits

Example 1

A 10-ft ladder rests against a vertical wall on level ground. The foot slides away at 2 ft/s. How fast is the top moving when the foot is 6 ft from the wall?

1. Let x be the foot's distance from the wall and y the top's height. The fixed ladder gives $x^2 + y^2 = 100$. At $x = 6$, $y = 8$ ft.
2. Differentiating in time gives $2x\left(dx/dt\right) + 2y\left(dy/dt\right) = 0$. Thus $dy/dt = -\left(6/8\right)\left(2\right) = -1.5$ ft/s.

The negative sign describes decreasing height. The length is fixed while both horizontal and vertical distances change.

Example 2

Evaluate $\lim_{x \to 0} \left[e^{2x} - 1 - 2x\right]/x^2$.

1. Substitution gives $0/0$. The functions are differentiable and the denominator derivative $2x$ is nonzero near, but away from, zero.
2. The ratio of first derivatives is $\left(2e^{2x} - 2\right)/\left(2x\right)$, still $0/0$. Apply the rule again to obtain the limit of $4e^{2x}/2$, which is 2.

The ratio after the first use still needs a limit. Check the new form before another differentiation.

> **Common error.** A negative acceleration tells you that velocity is decreasing. If velocity is already negative, its magnitude is increasing and the particle is speeding up.

LESSON 06 / CED 4.1

Interpreting a derivative in context

Explain a derivative at a specified input, including its sign and units.

CHA-3.A • OpenStax Calculus, Vol. 1, §3.4

Read $Q'(a)$ as an instantaneous change in the output Q per unit change in the input near a. Its units are output units divided by input units. A negative derivative describes a falling output as the input increases. The value $Q(a)$ tells how much is present. A single derivative does not establish an exact change across an entire interval.

Worked example 1

Elevation $E(x)$ is measured in meters at x kilometers along a trail. Interpret $E(3) = 810$ and $E'(3) = -25$.

1. At kilometer 3 the trail elevation is 810 m.
2. Near that point the elevation is falling at 25 meters per kilometer. Its exact elevation one kilometer later also depends on the intervening slopes.

Worked example 2

A pump's inflow rate $R(t)$ is measured in liters per minute at minute t. Interpret $R(4) = 7$ and $R'(4) = 0.6$.

1. Four minutes after measurement began, water enters at 7 L/min.
2. The inflow rate is increasing at 0.6 L/min per minute, or 0.6 L/min^2. The output being differentiated is itself a rate.

Check your understanding

4.1 C 1 A sensor reports $T'(8) = -0.4$ degrees Celsius per minute. Could $T(8)$ be positive?

4.1 C 2 Ticket demand $N(p)$ locally falls by four tickets per additional dollar near price $p = 30$. Express this as a derivative.

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.2

Position, velocity, acceleration, and speed

Use velocity and acceleration to distinguish direction of travel from changes in speed.

CHA-3.B • OpenStax Calculus, Vol. 1, §3.4

For motion along a line, $s(t)$ is signed position, $v = s'$ is signed velocity, speed is $|v|$, and $a = v'$ is acceleration. When velocity and acceleration have the same nonzero sign, speed increases. Opposite signs indicate decreasing speed. A zero velocity marks a time to inspect. The particle reverses only if the sign of velocity changes across it.

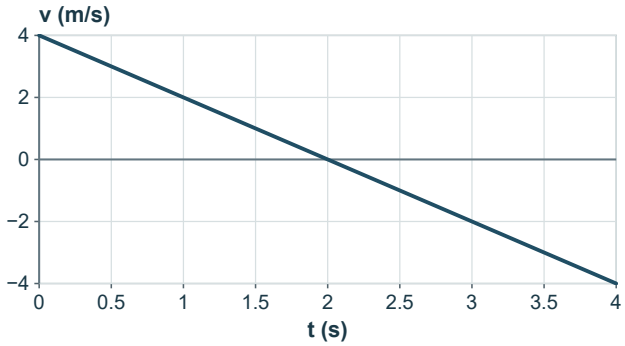
Worked example 1

A cart has $s(t) = t^3 - 6t^2 + 9t$ meters on $[0, 4]$ seconds. At $t = 2$, describe velocity, speed, and acceleration. Does it reverse at 3?

1. $v = 3(t-1)(t-3)$ m/s and $a = 6t - 12$ m/s^2.
2. At 2 s, $v = -3$ m/s, speed is 3 m/s, and acceleration is zero.
3. Velocity changes from negative to positive at 3 seconds, so the cart reverses.

Worked example 2

A runner's velocity graph is the line joining $(0, 4)$ and $(4, -4)$, with time in seconds and velocity in m/s. When is she speeding up?



1. The line's slope gives constant acceleration -2 m/s^2 in the interior.
2. Speed decreases while velocity is positive on $(0, 2)$, then increases on $(2, 4)$ as velocity becomes more negative.

Check your understanding

4.2 C 1 At time t seconds a cart's position is $s(t) = 9 - 5(t-2) - (t-2)^2$ meters. At $t = 2$, calculate its velocity and acceleration. What is the instantaneous rate of change of its speed?

4.2 C 2 A cart has velocity $v(t) = (t-2)^2$ m/s. Find its acceleration at $t = 1$ s and $t = 3$ s. Does it reverse direction at $t = 2$ s? Explain.

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.3

Rates beyond straight-line motion

Calculate and interpret rates in applied models, including derivatives of quantities that already represent rates.

CHA-3.C • OpenStax Calculus, Vol. 1, §3.4

Identify the quantity that changes and the input with respect to which it changes. A derivative with respect to time gives a rate per unit time. A derivative with respect to length or price instead gives a rate per unit of length or price. For table data, use a nearby secant to estimate a derivative and state its units. The measurements do not determine the exact instantaneous rate between them.

Worked example 1

Greenhouse moisture is $M(t) = 30 + 4t - t^2/2$ kg on $0 \le t \le 6$ hours. When does it stop increasing? Interpret M''.

1. $M' = 4 - t$ kg/h is positive before 4 hours and negative afterward. Moisture stops increasing at 4 hours.
2. $M'' = -1$ kg/h^2: the moisture-change rate decreases by 1 kg/h each hour.

Worked example 2

Pump delivery rates at minutes 4, 5, and 6 are 11.2, 12.1, and 12.6 L/min. Estimate how fast the delivery rate changes at minute 5.

1. The bracketing secant slope is $(12.6 - 11.2)/(6 - 4) = 0.7$ L/min^2.
2. The delivery rate is increasing at an estimated 0.7 L/min per minute. The table supplies a local estimate.

Check your understanding

4.3 C 1 A smooth cost model for q booklets is $C(q) = 50 + 1.6q + 0.002q^2$ dollars. Interpret $C'(100)$.

4.3 C 2 At outdoor temperatures of 18 and 22 degrees Celsius a sensor draws 14 and 10 watts. Estimate how power draw changes per degree near 20.

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.4

Build a related-rates equation

Choose variables with a shared input and derive a relationship between their rates.

CHA-3.D • OpenStax Calculus, Vol. 1, §4.1

Write an equation that continues to hold as the measurements change. Label fixed lengths and quantities that depend on time. Differentiate both sides with respect to time, attaching a time derivative to each changing quantity. A snapshot tells where to evaluate that new equation. Substituting current dimensions too early can remove a changing variable from the calculation.

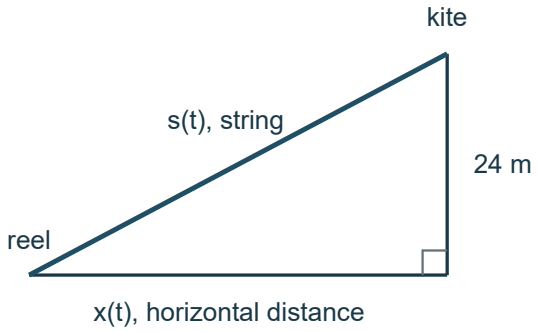
Worked example 1

A rectangular tarp has changing length L and width W. Derive its area rate. Why would $L'W'$ have incorrect units?

1. Both factors in $A = LW$ may change, giving $A' = L'W + LW'$.
2. With lengths in meters and time in seconds, both terms are in m^2/s. The product $L'W'$ is in m^2/s^2.

Worked example 2

A kite flies at a fixed height of 24 m. The horizontal distance x from the reel changes, as does the taut string length s. Relate x' to s'.



Fixed height; x and s change

1. The horizontal leg $x(t)$ and string length $s(t)$ change with time; the vertical leg stays 24 m. The labeled right triangle gives $s^2 = x^2 + 24^2$.
2. Differentiate before substituting a snapshot: $2ss' = 2xx'$, so $s' = (x/s)\,x'$. The length ratio is dimensionless. For $x > 0$, moving away from the reel makes both rates positive.

Check your understanding

4.4 C 1 Differentiate a circle's circumference and area in time for changing radius r. Which rate needs the current radius as well as r'?

4.4 C 2 A cylinder holds water at depth h cm and has fixed radius 4 cm. At a certain time $h' = 0.3$ cm/s. Calculate V' and explain why the formula needs no radius-rate term.

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.5

Solve related-rates problems

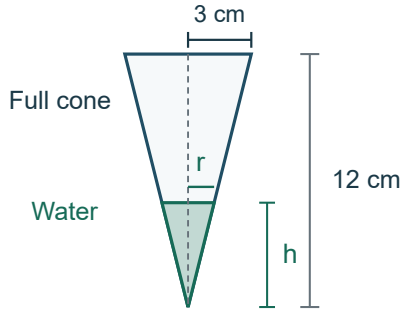
Combine a geometric constraint and measured rates to find an unknown instantaneous rate.

CHA-3.E • OpenStax Calculus, Vol. 1, §4.1

Sketch the changing figure. Locate a fixed relationship, often similarity or the Pythagorean theorem, that lets you eliminate a variable. Differentiate before inserting the dimensions at the selected instant. Translate wording such as “draining” into a signed rate, or “diameter” into twice a radius. Report the answer with units and a direction.

Worked example 1

A conical funnel points down, 12 cm high with rim radius 3 cm. Water enters at 6 cm^3/s. Find the depth rate when depth is 4 cm.



1. The full cone and the smaller water cone have the same apex angle. Their corresponding radii and heights satisfy $r/h = 3/12$, so $r = h/4$ for positive water depth.
2. Eliminate the changing radius before differentiating: $V = \pi r^2 h/3 = \pi h^3/48$. Then $V' = \left(\pi h^2/16\right) h'$.
3. At $h = 4$ cm, $6 = \pi h'$. Thus the water surface rises at $6/\pi$ cm/s. The surface area there is π cm^2, which explains why dividing volume rate by it gives the depth rate.

LESSON 06 / CED 4.5 / CONTINUED

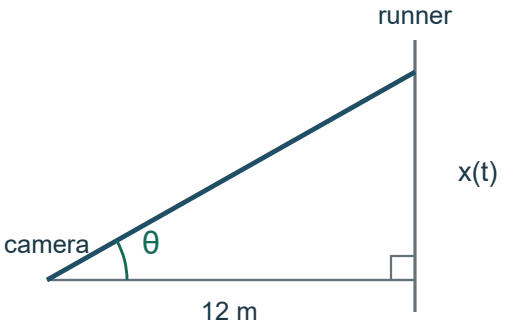
Solve related-rates problems

Combine a geometric constraint and measured rates to find an unknown instantaneous rate.

CHA-3.E • OpenStax Calculus, Vol. 1, §4.1

Worked example 2

A camera is 12 m from a straight track. A runner is x meters along the track from its closest point. When $x = 5$ and $x' = 4$ m/s, find the rate of change of the camera angle θ from the shortest sightline.



θ is measured in radians

1. From the triangle, $\tan\theta = x/12$, so $\sec^2\theta\,\theta' = x'/12$.
2. $\sec^2\theta = 1 + (5/12)^2 = 169/144$. Therefore $\theta' = (4/12)(144/169) = 48/169$ rad/s in the increasing-angle direction.

Check your understanding

4.5 C 1 A sphere has radius 2 cm and volume growing at 8π cm^3/s. Find the radius rate. Interpret its sign by naming what moves, in which direction, and with what units.

4.5 C 2 A 6 m by 8 m rectangle's 6 m length shrinks at 0.25 m/min while its width grows at 0.5 m/min. What happens to its area?

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.6

Local linearity and tangent estimates

Construct a local linear model and use concavity to judge the direction of its error.

CHA-3.F • OpenStax Calculus, Vol. 1, §4.2

The tangent approximation at a is $L(x) = f(a) + f'(a)(x - a)$. The slope times the input change gives a predicted output change. Add the starting value to estimate the new output. Concave-up graphs lie above their tangent on an interval of concavity; concave-down graphs lie below. The slope at one point alone cannot determine error direction, and a distant estimate can be poor.

Worked example 1

Estimate $\sqrt[3]{7.7}$ using $x^{1/3}$ at 8. Is the estimate high or low?

1. $f(8) = 2$ and $f'(8) = 1/12$, so $L(x) = 2 + (x - 8)/12$.
2. $L(7.7) = 1.975$. Since $f''(x) = -2/\left(9x^{5/3}\right) < 0$ on the interval, the tangent estimate is high.

Worked example 2

At minute 20 a tank contains 34 L and its net inflow is 1.5 L/min. Its net inflow increases throughout minutes 20 to 21. Estimate the amount at 20.4 min and judge the error.

1. The tangent estimate for volume is $34 + 1.5(0.4) = 34.6$ L.
2. Volume's derivative is net inflow. It increases, making the volume graph concave up, so the estimate is low.

Check your understanding

4.6 C 1 Given $f(5) = 9$ and $f'(5) = -3$, estimate $f(4.9)$ and identify the predicted change.

4.6 C 2 Given only $g(1) = 2$ and $g'(1) = 4$, use the tangent line to estimate $g(1.1)$. Do these data determine whether the estimate is high or low?

Explain your reasoning before turning to the check key.

LESSON 06 / CED 4.7

Limits of indeterminate quotients

Determine whether L'Hôpital's rule applies, then use it to evaluate a qualifying limit.

LIM-4.A • OpenStax Calculus, Vol. 1, §4.8

Check numerator and denominator limits separately. The forms $0/0$ and ∞/∞ (including signed infinities) can justify L'Hôpital's rule. The functions must be differentiable and the denominator derivative nonzero on a punctured neighborhood or tail. If f'/g' has a finite or infinite limit there, f/g has the same limit. The rule compares limits, not values of those quotients at ordinary inputs. Check the form again before a second application.

Worked example 1

Evaluate $\lim_{x \to 0} \frac{\ln(1+4x)}{e^{2x}-1}$. Justify the choice of method.

1. Both numerator and denominator tend to zero and $2e^{2x}$ stays positive nearby.
2. The derivative ratio is $\frac{4/(1+4x)}{2e^{2x}}$, whose limit is 2.

Worked example 2

Determine $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$. Do some early rising values settle the limit?

1. Both numerator and denominator tend to infinity, and the denominator derivative is positive.
2. The derivative ratio $(1/x)/\left(1/\left(2\sqrt{x}\right)\right) = 2/\sqrt{x}$ tends to zero. The original ratio therefore tends to zero regardless of its early increase.

Check your understanding

4.7 C 1 A student differentiates top and bottom of $\lim_{x \to 0} (2+x)/(3+x)$ and gets 1. Explain the error. Then evaluate $\lim_{x \to 0} \left(e^{3x}-1\right)/\sin(2x)$, checking the conditions for your method.

4.7 C 2 Evaluate $\lim_{x \to \infty} x/e^{0.5x}$ and check the initial form.

Explain your reasoning before turning to the check key.

LESSON 07

Reading a function through its derivatives

Use derivative evidence to make and justify claims about a graph, even when you do not have its formula.

CED 5.1–5.9 • OpenStax Calculus Vol. 1, sections 4.3–4.5

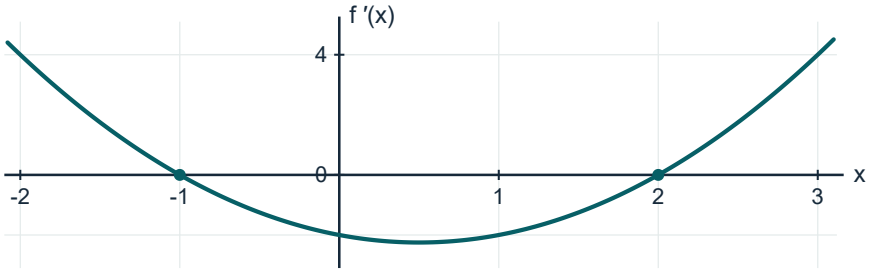
The idea

If f' is positive throughout an interval, f increases there. If f' is negative throughout, f decreases. Inside the interval, a critical number is an input in the domain where $f' = 0$ or f' does not exist. Compare derivative signs on both sides to classify an interior critical number. Check endpoints separately in a closed interval.

An increasing f' means that slopes get larger and f is concave up. Slopes can get larger while still negative. A point on a continuous graph is an inflection point when the concavity changes across it. Use signs of f'' or trends in f' to establish the change.

When $f'(c) = 0$, a positive $f''(c)$ gives a local minimum and a negative $f''(c)$ gives a local maximum. If $f''(c) = 0$, try a sign chart for f' instead. For an absolute extremum on a closed interval, compare function values at the endpoints and every interior critical number.

The Extreme Value Theorem guarantees an absolute maximum and minimum for a function continuous on a closed interval. The Mean Value Theorem also requires differentiability on the open interval. It guarantees some interior tangent with slope equal to the secant slope between the endpoints.



The vertical coordinate is the derivative of f. The curve crosses the horizontal axis at $x = -1$ and $x = 2$.

On the displayed interval, where does f increase and decrease? Classify the two interior critical points.

LESSON 07 / WORKED EXAMPLES

Reading a function through its derivatives

Example 1

On $[-2, 3]$, find the absolute extrema of $f(x) = x^3 - 3x$.

1. The polynomial is continuous on the closed interval, so absolute extrema exist. Solve $f'(x) = 3(x-1)(x+1) = 0$ to get $x = -1, 1$.
2. Evaluate all candidates: $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, and $f(3) = 18$.
3. The absolute maximum is 18 at $x = 3$. The absolute minimum is -2 at both $x = -2$ and $x = 1$.

Classifying the interior turns alone would miss the largest value and one location of the smallest value.

Example 2

A differentiable function has derivative $g'(x) = (x-2)^2$ on the real line. Describe g on each side of 2.

1. The derivative is positive on both sides of 2. The graph rises on both sides and has a horizontal tangent at 2, so there is no extremum there.
2. Since $g''(x) = 2(x-2)$ changes from negative to positive, the graph is concave down before 2 and concave up after 2.
3. The point $\big(2, g(2)\big)$ is a stationary inflection point.

The sign of the first derivative answers a different question from the sign of the second derivative.

> **Common error.** A zero second derivative gives a candidate for an inflection point. For $f(x) = x^4$, $f''(0) = 0$ but the graph is concave up on both sides.

Explain it aloud

If a graph is going downhill but flattening out, what can you say about the first derivative and how it is changing?

LESSON 07 / CED 5.1

A secant slope that a tangent must match

Check whether the Mean Value Theorem guarantees an instantaneous rate, then interpret or locate it.

FUN-1.B • OpenStax Calculus, Volume 1, §4.4

Draw the secant through the two endpoints and compute its slope. If f is continuous on $[a, b]$ and differentiable on (a, b), then some c strictly between the endpoints satisfies $f'(c) = \frac{f(b) - f(a)}{b - a}$. A corner at an endpoint is harmless. A corner inside the interval can prevent this conclusion. The theorem guarantees a matching tangent slope without telling you how many such points exist.

Worked example 1

Find every point where the tangent to $f(x) = x^3$ is parallel to the secant through $x = -2$ and $x = 2$.

1. A polynomial is continuous and differentiable throughout the interval.
2. The secant slope is $\frac{8 - (-8)}{2 - (-2)} = 4$. Solve $3c^2 = 4$.
3. Both $c = -2/\sqrt{3}$ and $c = 2/\sqrt{3}$ lie inside $(-2, 2)$. Each tangent has slope 4.

Worked example 2

A pressure sensor reads 34 kPa at 2 seconds and 52 kPa at 8 seconds. Pressure is continuous and differentiable throughout. Does it rise at exactly 3 kPa/s at any instant?

1. The average rate is $(52 - 34) / (8 - 2) = 3$ kPa/s.
2. The stated hypotheses allow the MVT on $[2, 8]$. At least once strictly between the readings, the instantaneous rate is 3 kPa/s.

Check your understanding

5.1 C 1 What do equal endpoint values imply if both MVT hypotheses hold?

5.1 C 2 A graph has a corner at $x = 3$ inside $[1, 5]$. Can you conclude no tangent matches the secant?
Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.2

Absolute extrema, local extrema, and critical numbers

Use the Extreme Value Theorem and distinguish a possible extremum from an attained one.

FUN-1.C • OpenStax Calculus, Volume 1, §4.3

Continuity on a closed, bounded interval guarantees both a largest value and a smallest value, each attained at least once. More than one input can attain either extreme. A local extremum compares a value with nearby values; an absolute extremum compares it with every value in the domain. Interior critical numbers are inputs in the domain where $f'=0$ or f' does not exist. Include them and the endpoints in a search for absolute extrema. A critical number may have no extremum.

Worked example 1

For $f(x)=|x-1|$ on $[-1,3]$, locate all points attaining either absolute extreme. Does the cusp prevent using the theorem?

1. Absolute value is continuous on this closed interval, so the theorem applies. Differentiability is unnecessary.
2. Distance to 1 is zero only at 1 and at most 2. The minimum is 0 at $x=1$; the maximum is 2 at both $x=-1$ and $x=3$.

Worked example 2

For $g(x)=x^3$ on $[-1,2)$, which extremes are attained? Does the critical number give a local extremum?

1. The missing endpoint prevents an EVT guarantee. The cubic strictly increases, so its minimum is -1 at the included endpoint.
2. Values approach 8 but every input has a larger input still in the domain. No maximum is attained. At zero the slope is zero but nearby values lie on both sides of zero.

Check your understanding

5.2 C 1 Compare two functions that agree with $f(x)=x$ near 1 except at input 1. In the first, $f(1)$ is undefined. In the second, set $f(1)=10$. Is 1 a critical number in either case? Explain.

5.2 C 2 A function is continuous on $[2,5]$, but no formula or graph is supplied. What can you guarantee about its largest and smallest values? Could either occur at more than one input?

5.2 C 3 For $g(x)=2+\sqrt{4-x^2}$ on $[-2,2]$, explain why the EVT applies. Find the absolute maximum and minimum values and every input where each occurs.

Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.3

Intervals of increase and decrease

Translate derivative signs into intervals of monotonic behavior, taking account of domain restrictions.

FUN-4.A • OpenStax Calculus, Volume 1, §4.5

Read the derivative graph's height relative to zero. Where $f' > 0$ throughout an interval, f increases; where $f' < 0$, it decreases. For a formula, split the domain at derivative zeros, points where the derivative fails to exist, and gaps in the original domain. Test signs between boundaries. If f' vanishes at an isolated point but keeps the same sign, the original function may remain strictly monotone across that point.

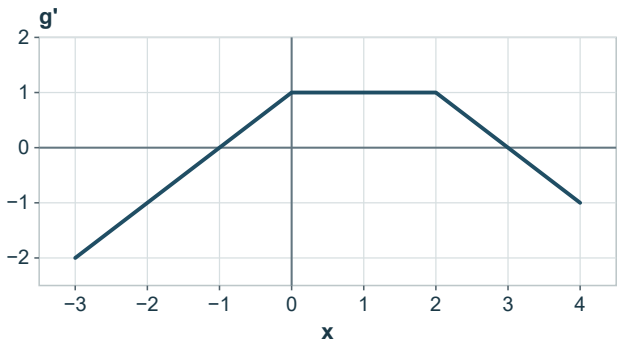
Worked example 1

Describe f if $f'(x) = (x+3)(x-1)^2$ on the real line.

1. Candidates are -3 and 1. Away from these numbers, the square is positive so the sign comes from $x+3$.
2. Thus f decreases on $(-\infty, -3)$ and increases on $(-3, \infty)$. The horizontal tangent at 1 does not interrupt that increase.

Worked example 2

The graph is $y = g'(x)$. Where does g increase? Is it increasing everywhere its derivative graph slopes upward?



1. The derivative crosses zero at -1 and 3. It is positive between the zeros, so g increases on $(-1, 3)$.
2. On $(-3, -1)$, its derivative graph rises but remains below zero. The function g decreases there.

Check your understanding

5.3 C 1 If $f'(x) = -4$ throughout $(2, 6)$, how does f behave?

5.3 C 2 Can a function decrease near a point where its derivative graph slopes upward?
Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.4

The first derivative test

Classify interior critical points using first derivative signs on the two adjacent intervals.

FUN-4.A • OpenStax Calculus, Volume 1, §4.5

Assume the function is continuous at the candidate. If f' changes from positive to negative, the function rises and then falls, producing a local maximum. A negative-to-positive change produces a local minimum. A sign that stays the same gives no turn. The test works at a corner even when the point itself has no derivative.

Worked example 1

A continuous function has $f'(x) = (x+2)^2(1-x)$. Classify its critical numbers.

1. The derivative vanishes at -2 and 1. Its signs on the three consecutive intervals are $+, +, -$.
2. The function increases on both sides of -2, so there is no extremum there. At 1 the positive-to-negative change gives a local maximum.

Worked example 2

Classify the critical point of $g(x) = 3 - |x-4|$.

1. For $x < 4$, $g = x - 1$ and $g' = 1$. For $x > 4$, $g = 7 - x$ and $g' = -1$.
2. Continuity and the positive-to-negative sign change give a local maximum at $(4, 3)$, even though the slope is undefined there.

Check your understanding

5.4 C 1 A continuous f has $f'(7) = 0$ and negative derivative on both sides nearby. Is 7 a local minimum?

5.4 C 2 The derivative graph crosses upward through the horizontal axis at 2. What happens on the original graph?

Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.5

Comparing candidates for absolute extrema

Find all eligible inputs, compare function values, and report both the absolute extreme values and their locations.

FUN-4.A • OpenStax Calculus, Volume 1, §4.3

On a closed interval where f is continuous, absolute extrema must occur at an endpoint or an interior critical number. List all of those inputs first. Evaluate the original function at each, then compare the outputs. An input that gives a horizontal tangent is a candidate even if it fails to produce a local turn. Include any interior point in the domain where the derivative fails to exist.

Worked example 1

Find both absolute extremes of $f(x) = x^3 - 12x$ on $[-3, 3]$.

1. Solve $f' = 3x^2 - 12 = 0$ to obtain $-2, 2$. Together with endpoints, the inputs are $-3, -2, 2, 3$.
2. The corresponding values are $9, 16, -16, -9$. The absolute maximum is 16 at -2, and the absolute minimum is -16 at 2.

Worked example 2

A continuous curve on $[-2, 5]$ has exactly two interior critical numbers: a corner at 0 and a horizontal tangent at 3. Its values at $-2, 0, 3, 5$ are $4, -1, 4, 2$. Where are the absolute extrema?

1. All four inputs are candidates. A corner counts even without a derivative, and both endpoints must be checked.
2. The absolute minimum is -1 at 0; the maximum is 4, attained at both -2 and 3.

Check your understanding

5.5 C 1 Why can a sign change that proves a local maximum be insufficient to prove an absolute maximum?

5.5 C 2 On $[-2, 3]$, a continuous function has a single interior critical number at 1. Its values at $-2, 1, 3$ are $7, -1, 8$. Identify the absolute extrema and explain why this list suffices.

Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.6

Concavity and changes of concavity

Use changing tangent slopes or second derivative signs to locate concavity intervals and justify inflection points.

FUN-4.A • OpenStax Calculus, Volume 1, §4.5

A concave-up graph has increasing tangent slopes. A concave-down graph has decreasing tangent slopes. A positive second derivative throughout an interval is sufficient for concavity up, and a negative second derivative is sufficient for concavity down. The second derivative may vanish at isolated points without changing concavity. For an inflection point, the function must be continuous there and the concavity must change across it. Test both sides of zeros and undefined values of the second derivative. A point missing from the graph cannot be an inflection point.

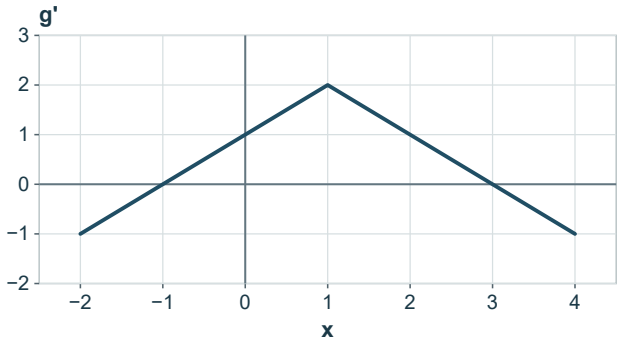
Worked example 1

Find the intervals of concavity and the coordinates of the inflection points of $f(x) = x^4 - 4x^3$.

1. $f''(x) = 12x(x-2)$. Its signs across 0 and 2 are $+, -, +$.
2. The graph is concave up on $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$. Both changes occur at points of continuity: $(0, 0)$ and $(2, -16)$.

Worked example 2

From the graph of g', identify where g changes concavity.



1. The slopes of g rise through $x = 1$ and then fall. Thus g is concave up on $(-2, 1)$ and down on $(1, 4)$.
2. Since g is differentiable there, it is continuous. The change at 1 gives an inflection point even though g' has a corner there.

Check your understanding

5.6 C 1 Does $f''(c) = 0$ alone prove an inflection point? Use $f(x) = x^4$ to decide.

5.6 C 2 Can f be increasing and concave down? Describe what the graph of f' would do.
Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.7

The second derivative test

Classify a stationary point by its curvature and select another test when curvature at the point gives no answer.

FUN-4.A • OpenStax Calculus, Volume 1, §4.5

Start by checking $f'(c) = 0$. For a sufficiently smooth function near c, a positive second derivative at c gives a local minimum and a negative second derivative gives a local maximum. The slope comparison explains the test: when slopes are increasing through zero, the graph falls before it rises. If $f''(c) = 0$ or is undefined, this test is inconclusive. Use a sign change in f' or compare nearby function values. On an interval where the function is continuous and has just one interior critical number, a local maximum there is also the absolute maximum. The corresponding statement holds for a local minimum.

Worked example 1

Classify all stationary points of $f(x) = x^3 - 3x^2 - 9x$.

1. $f' = 3(x-3)(x+1)$ gives $c = -1$ and $c = 3$. The second derivative is $6x - 6$.
2. At -1, $f'' = -12$, giving a local maximum at $(-1, 5)$. At 3, $f'' = 12$, giving a local minimum at $(3, -27)$.

Worked example 2

The functions $u(x) = x^4$ and $v(x) = x^3$ both have first and second derivatives equal to zero at the origin. Classify the origin for each.

1. The second derivative test gives no classification for either function.
2. For u, values are positive on both sides of zero, so zero is a local minimum. In fact, $u(x) = x^4 \geq 0$ for every real x, making this an absolute minimum as well. The continuous function has only one critical input, 0; the direct value comparison confirms the global conclusion.
3. For v, the graph increases through zero and has no extremum there. One critical input alone does not guarantee an absolute extremum: that input must first be shown to give a local extremum.

Check your understanding

5.7 C 1 If $f'(4) = 2$ and $f''(4) < 0$, is there a local maximum at 4?

5.7 C 2 A graph has a horizontal tangent at 1 and $f''(1) = 6$. How can you describe the point precisely?
Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.8

Sketching from slopes and curvature

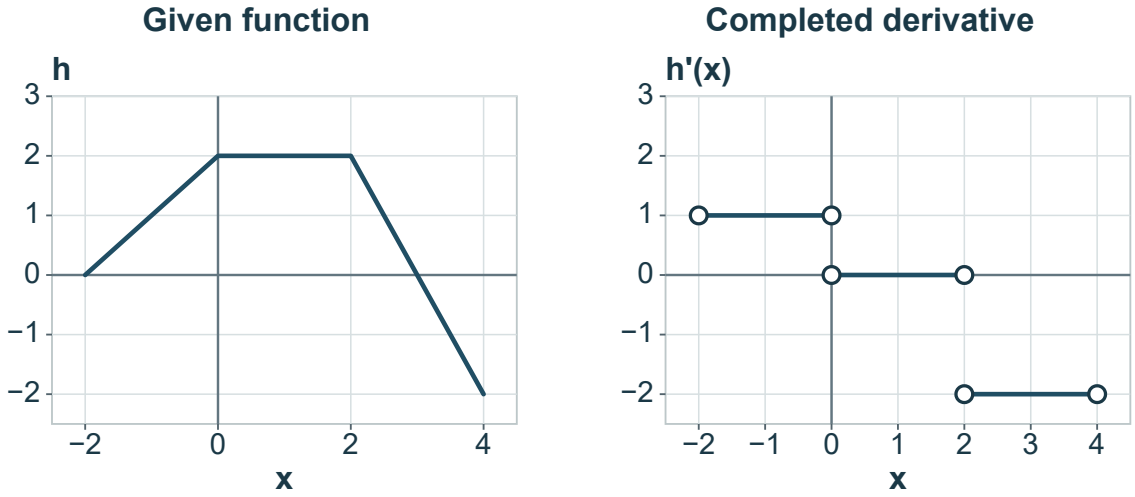
Sketch a function or its derivative from graphical or numerical evidence and preserve the features the evidence determines.

FUN-4.A • OpenStax Calculus, Volume 1, §§3.2, 4.5

To sketch f from f', mark where slopes are positive, zero, or negative. Then mark where slopes get larger or smaller so your sketch has the right curvature. An initial value anchors the height. In the other direction, read tangent slopes from a graph of f and plot those slopes as heights on the graph of f'. A straight segment gives a constant derivative; a corner with unequal one-sided slopes leaves a gap in the derivative graph.

Worked example 1

The left graph shows h. Sketch h' on the interior of the displayed domain. The completed sketch at right shows how each straight segment becomes a constant slope.



1. From $(-2, 0)$ to $(0, 2)$ the segment has slope 1. From $(0, 2)$ to $(2, 2)$ it is horizontal. From $(2, 2)$ to $(4, -2)$ its slope is -2.
2. Draw horizontal segments at heights $1, 0, -2$ on the three open intervals. Leave h' undefined at the corners 0 and 2.

LESSON 07 / CED 5.8 / CONTINUED

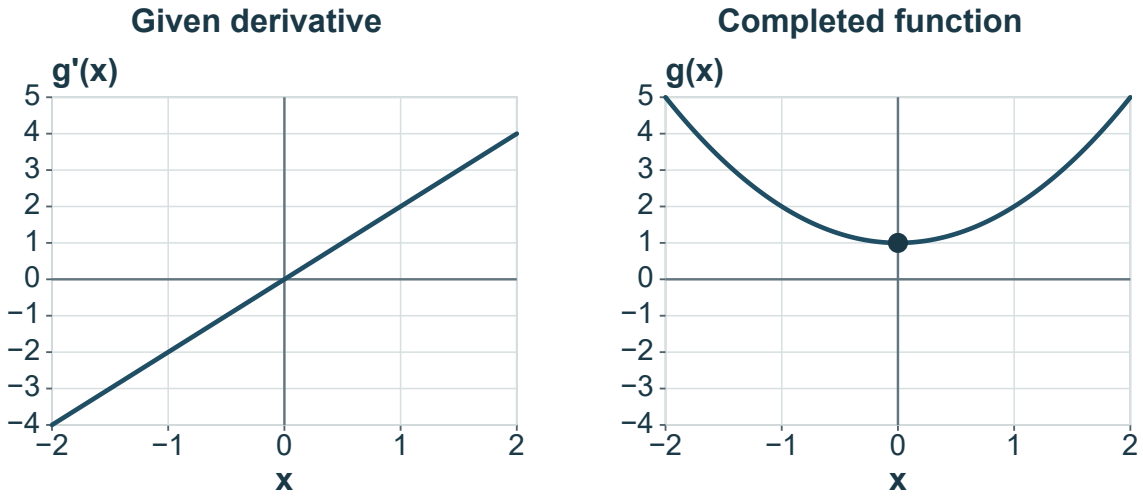
Sketching from slopes and curvature

Sketch a function or its derivative from graphical or numerical evidence and preserve the features the evidence determines.

FUN-4.A • OpenStax Calculus, Volume 1, §§3.2, 4.5

Worked example 2

A function has $g(0) = 1$ and $g'(x) = 2x$ on $[-2, 2]$. Sketch g and explain why its slope and height at 0 matter.



1. The derivative is negative to the left of zero and positive to the right. It increases throughout, giving a concave-up graph with a horizontal tangent at zero.
2. Place its minimum at $(0, 1)$. One way to check the entire sketch is to differentiate $g(x) = x^2 + 1$ and verify the given point.

Check your understanding

5.8 C 1 You have a complete graph of f' but no point on f. Which feature of your f sketch remains arbitrary?

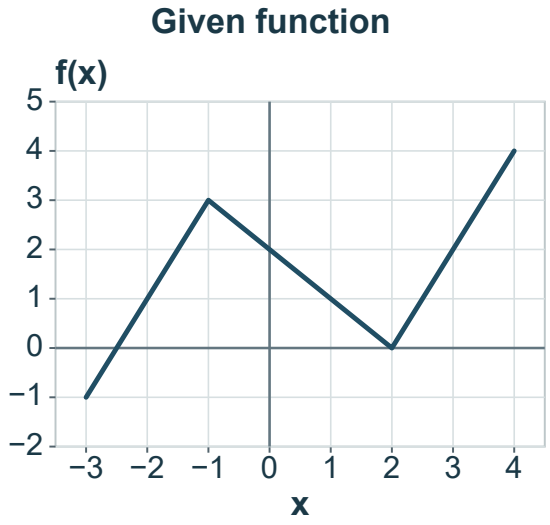
LESSON 07 / CED 5.8 / CONTINUED

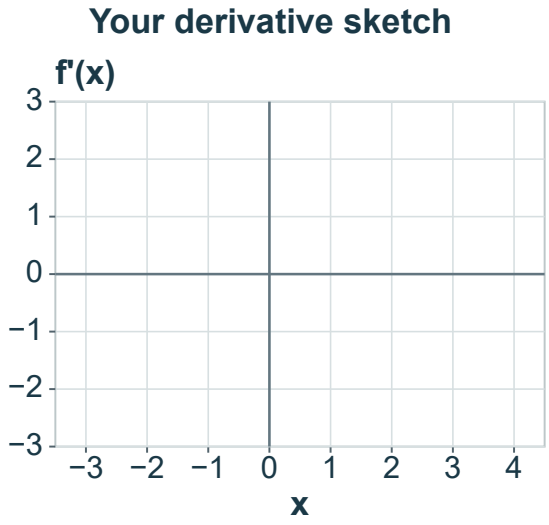
Sketching from slopes and curvature

Sketch a function or its derivative from graphical or numerical evidence and preserve the features the evidence determines.

FUN-4.A • OpenStax Calculus, Volume 1, §§3.2, 4.5

5.8 C 2 Use the left graph of f to sketch f' on the blank axes at right for $-3 < x < 4$. State the derivative on each interval and explain where it does not exist.





Explain your reasoning before turning to the check key.

LESSON 07 / CED 5.9

Connecting three graphs

Use the graphs of a function and its first two derivatives to justify behavior and reject incompatible graph assignments.

FUN-4.A • OpenStax Calculus, Volume 1, §4.5

Read each graph at the same horizontal position. A height on f' is a slope on f. A height on f'' is a slope on f'. Thus a positive but decreasing f' means f rises while bending down. A turn on f' can mark an inflection of f. A crossing of the horizontal axis by f' can mark an extremum of f. Sampled derivative values describe the given inputs; extend a claim across an interval only when the graph or formula supports it.

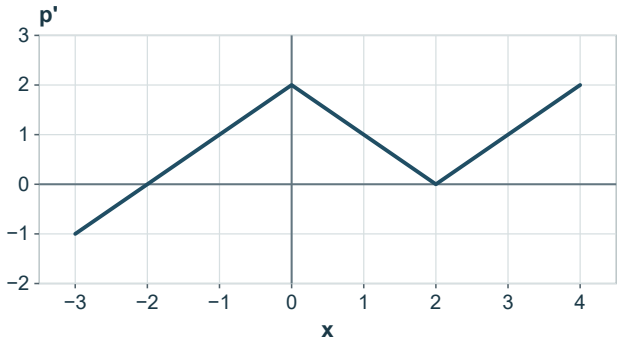
Worked example 1

At $x = 2$, a twice differentiable function has $f(2) = 7$, $f'(2) = -3$, and $f''(2) = 4$, with continuous second derivative. Describe the graphs near 2.

1. The point on f is $(2, 7)$ and its tangent slopes downward. Since f' is continuous and negative at 2, f decreases nearby.
2. The point on f' is $(2, -3)$. Its slope is 4; f'' stays positive nearby by continuity. Thus f' increases nearby and f bends upward while decreasing.

Worked example 2

The figure plots p'. Where does p have a local extremum? At which inputs does it have inflection points?



1. The derivative crosses upward through zero at -2, giving p a local minimum there. It touches zero at 2 but stays positive on both sides, so no extremum occurs there.
2. The derivative rises, falls, then rises, changing direction at 0 and 2. Thus p changes concavity at both 0 and 2.

Check your understanding

5.9 C 1 If f has a horizontal tangent and an inflection at the same input, must it also have an extremum there?

5.9 C 2 At a point, f' is above the axis and has negative slope. What can you say about f' and f'' at that point?

Explain your reasoning before turning to the check key.

LESSON 08

Constraints, optimal values, and curves

Turn a constraint into a one-variable model, establish a global optimum, and identify turns on an implicit curve.

CED 5.10–5.12 • OpenStax Calculus Vol. 1, sections 3.8, 4.5, 4.7

The idea

Draw the situation and label the quantities with units. Write the objective, the quantity you want to maximize or minimize. A separate constraint expresses how the dimensions depend on each other. Use it to eliminate one variable before taking a derivative.

State the feasible interval from the situation. On a closed interval compare the objective at endpoints and interior critical numbers. On an open interval a derivative sign change from positive everywhere before the candidate to negative everywhere after it proves a global maximum.

An equation in x and y can describe several branches. Differentiate both sides with respect to x, treating y as a function of x along a differentiable branch. Solve for y' and substitute a point from the original equation. To study concavity, differentiate the first derivative equation once more, keeping every factor of y' from the chain rule.

For a slope written as a quotient, solve for horizontal tangents where the numerator vanishes and the denominator stays nonzero. For a vertical tangent at a regular point, the denominator vanishes and the numerator stays nonzero. Every candidate must also satisfy the original curve equation.

> **Think it through.** Why must you know both the feasible interval and what each variable represents before you announce an optimal answer?

LESSON 08 / WORKED EXAMPLES

Constraints, optimal values, and curves

Example 1

A gardener has 30 m of fence for three sides of a rectangular garden adjoining a straight wall. Which dimensions maximize area?

1. Let w measure each side perpendicular to the wall and L the fenced side parallel to it. Then $L + 2w = 30$ and $A(w) = 30w - 2w^2$ for $0 < w < 15$.
2. Solve $A'(w) = 30 - 4w = 0$ to get $w = 7.5$. The derivative is positive before 7.5 and negative after it throughout the feasible interval.
3. The maximum area is 112.5 square meters, with $w = 7.5$ m and $L = 15$ m.

Using three fenced sides changes the constraint and therefore changes the maximizing dimensions.

Example 2

On $x^2 + 4y^2 = 16$, find horizontal tangents and describe the concavity of the upper branch.

1. Differentiating gives $2x + 8yy' = 0$, so $y' = -x/(4y)$ for $y \neq 0$. Horizontal tangents require $x = 0$; the curve gives points $(0, 2)$ and $(0, -2)$.
2. Differentiate the first equation again: $2 + 8(y')^2 + 8yy'' = 0$. Thus $y'' = -\left(1 + 4(y')^2\right)/(4y)$.
3. On the upper branch $y > 0$, the denominator is positive and the numerator is negative, so the branch is concave down.

The original equation gives the locations. The sign of the second derivative determines the bend of each branch.

> **Common error.** Setting the derivative to zero locates a candidate. A comparison of values or a sign argument over the whole feasible domain supplies the global claim.

LESSON 08 / CED 5.10

Building an optimization model

Identify a quantity to optimize, use a constraint to express it with one variable, and calculate its extreme value on the feasible domain.

FUN-4.B • OpenStax Calculus, Volume 1, §4.7

Name the objective and its units. Write a separate equation for the constraint, use it to eliminate another variable, and restrict the domain to physically possible choices. Then solve for critical inputs of the objective and give a global argument using endpoints or derivative signs. A critical input gives a design choice; substituting it back into the objective gives the optimal value.

Worked example 1

A right triangle has legs a, b with $a + b = 14$ cm. Find the greatest possible area.

1. The area is $A = ab/2$. Substitute $b = 14 - a$ to get $A(a) = a(14 - a)/2$ for $0 < a < 14$.
2. $A' = 7 - a$ is positive before 7 and negative after. The maximum is $49/2$ square centimeters, achieved when both legs are 7 cm.

Worked example 2

For $x > 0$, a model has cost $C(x) = 2x + 32/x$ dollars. Find its least cost.

1. The derivative $C' = 2 - 32/x^2$ vanishes at $x = 4$ in the given domain.
2. It is negative for $0 < x < 4$ and positive for $x > 4$. The global minimum cost is $C(4) = 16$ dollars.

Check your understanding

5.10 C 1 An open box has a square base of side x cm and height h cm. Its volume must be 32 cm³. Find the dimensions that minimize its material area, including the base and four sides. State the constraint, the one-variable objective, and why the minimum is global.

5.10 C 2 A rectangle has perimeter 12 cm. Let x be one side. Write a one-variable area model, state its physical domain, and use a derivative to find the maximum area.

Explain your reasoning before turning to the check key.

LESSON 08 / CED 5.11

Interpreting optimal values in context

Distinguish the optimal input from the optimal output and interpret how a physical or business constraint changes the answer.

FUN-4.C • OpenStax Calculus, Volume 1, §4.7

State what each variable measures before differentiating. After choosing a global optimizer, report the design or time, the quantity it achieves, and the units of both. If the unconstrained optimizer is unavailable, use the actual feasible domain. When an input is restricted to whole numbers and the objective increases up to its continuous maximizer and decreases afterward, the two adjacent feasible integers suffice. For a minimum, reverse those monotonic directions. Compare the objective values at both neighbors. Without this single-turn behavior, also examine other critical regions and domain boundaries.

Worked example 1

A clinic's daily modeled cost per sample is $c(n) = n + 144/n$ dollars for whole-number batches of 5 to 10 samples. What batch size gives the lowest cost per sample?

1. In the continuous model $c' = 1 - 144/n^2$, zero at 12. That input exceeds the batch limit.
2. The derivative is negative throughout $(5, 10)$, so cost falls through the feasible range. Use 10 samples per batch. The cost is $10 + 144/10 = 24.40$ dollars per sample.

Worked example 2

An instrument records signal strength $S(t) = 9 + 8t - t^2$ for $1 \le t \le 6$ minutes. Report the strongest signal, its time, and why it is the strongest across this recording.

1. $S' = 8 - 2t$ changes sign from positive to negative at 4. Thus the model increases until 4 and decreases afterward.
2. At 4 minutes, the instrument reaches its maximum reading of 25 signal units. Values at 1 and 6 minutes are 16 and 21, respectively.

Check your understanding

5.11 C 1 A water-use model attains its minimum at a valve setting of 6 with output 11 L/min. Which number answers 'what is the least water-use rate?'

5.11 C 2 A model has a maximum at 7.4 when the input can only be a whole number. Is ordinary rounding always justified?

Explain your reasoning before turning to the check key.

LESSON 08 / CED 5.12

Critical points and curvature of implicit curves

Find critical points on an implicit curve and use first and second derivatives to justify local behavior of its branches.

FUN-4.D / FUN-4.E • OpenStax Calculus, Volume 1, §§3.8, 4.5

Differentiate the defining equation with respect to x and isolate y'. For a regular point, a zero numerator with a nonzero denominator gives a horizontal tangent. A zero denominator with a nonzero numerator gives a vertical tangent. In both cases solve the condition together with the original equation. If numerator and denominator both vanish, inspect the local geometry separately. To find concavity on a differentiable branch, differentiate again, remembering that y and y' depend on x.

Worked example 1

For $x^2 + xy + y^2 = 3$, locate its horizontal and vertical tangents. Classify the two horizontal tangent points by curvature.

1. $y' = -\left(2x + y\right) / \left(x + 2y\right)$. Horizontals require $y = -2x$. Substitution gives $3x^2 = 3$, hence $(-1, 2)$ and $(1, -2)$. Verticals require $x = -2y$, giving $(-2, 1)$ and $(2, -1)$. The opposite factor is nonzero at each.
2. A second differentiation gives $\left(x + 2y\right) y'' = -2 - 2y' - 2\left(y'\right)^2$. At the horizontal points, $y'' = -2/3$ at $(-1, 2)$ and $2/3$ at $(1, -2)$. The former is a local maximum of its branch and the latter a local minimum.

Worked example 2

For the sideways parabola $y^2 = x + 4$, describe the two branches for $x > -4$ and identify the tangent at their meeting point.

1. For $y \neq 0$, $y' = 1/\left(2y\right)$. The upper branch increases and the lower decreases. At $(-4, 0)$ the tangent is vertical; using $x = y^2 - 4$ gives $dx/dy = 0$ there.
2. Differentiate $2yy' = 1$ to get $y'' = -\left(y'\right)^2 / y = -1/\left(4y^3\right)$. The upper branch is concave down and the lower is concave up.

LESSON 08 / CED 5.12 / CONTINUED

Critical points and curvature of implicit curves

Find critical points on an implicit curve and use first and second derivatives to justify local behavior of its branches.

FUN-4.D / FUN-4.E • OpenStax Calculus, Volume 1, §§3.8, 4.5

Check your understanding

5.12 C 1 For $x^2 - y^2 = 0$, the ratio $y' = x/y$ gives $0/0$ at the origin. Does that prove a vertical tangent?

5.12 C 2 For $x^2 + y^2 = 5$, find y'' at $(1, 2)$ and $(1, -2)$ by differentiating the relation twice. State the concavity of each branch near its point.

5.12 C 3 For the curve $x = 1 + y^2$, compute dy/dx wherever it exists and find the vertical tangent. Why does a zero denominator alone fail to make $(0, 0)$ a candidate?

5.12 C 4 For the branch of $xy = 4$ through $(2, 2)$, find the first and second derivatives at the point. Use them to describe the branch's direction and concavity nearby.

5.12 C 5 On $y^2 + x = 4$, locate the point with a vertical tangent and verify the tangent direction using x as a function of y.

5.12 C 6 For the relation $y^2 + 4x^2 = 4$, determine every horizontal-tangent point. Check the denominator when you use a slope formula.

Explain your reasoning before turning to the check key.

LESSON 09

From rate samples to a definite integral

Build and interpret sums of rate times interval width, then connect their limit to signed accumulation.

CED 6.1–6.3, 6.6 • OpenStax Calculus Vol. 1, sections 5.1–5.2; Vol. 2, section 3.6 (trapezoids)

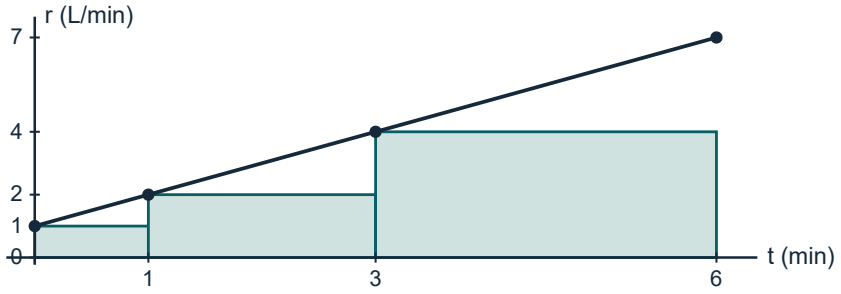
The idea

A flow of 5 liters per minute for 2 minutes adds 10 liters. For a varying rate, break time into short intervals, use a representative rate on each, and add rate times duration. The product of the vertical and horizontal units gives the units of the integral.

Left, right, and midpoint sums sample a rate at the specified location in each interval. Trapezoids use the mean of the endpoint rates. Use each interval's actual width. For a nondecreasing function the left sum is at most the integral, and the right sum is at least the integral. For a concave-up function, trapezoidal chords lie on or above its graph. A straight segment gives the exact integral on that panel.

For n equal subintervals of $[a, b]$, the width is $\Delta x = (b - a)/n$. A right endpoint is $x_i = a + i\Delta x$. For continuous f, the limit of $\sum_{i=1}^{n} f(x_i)\,\Delta x$ as n grows is $\int_a^b f(x)\,dx$. The choice of left, right, or midpoint yields the same limit.

Integrals add over adjacent intervals and distribute over sums and constants. Reversing the bounds reverses the sign. Area below the horizontal axis contributes negatively. A signed result can be negative even when the total geometric area is positive.



The graph gives inflow rate $r(t) = t + 1$ in liters per minute. Shaded rectangles use left endpoints on $[0, 1]$, $[1, 3]$, and $[3, 6]$.

Estimate the amount that enters in six minutes. Explain the direction of the error.

LESSON 09 / WORKED EXAMPLES

From rate samples to a definite integral

Example 1

A tank fills at rates of 2, 5, and 7 L/min at minutes 0, 1, and 4. Estimate the added water with trapezoids.

1. Use two intervals of widths 1 and 3 minutes. Their mean endpoint rates are $(2+5)/2 = 3.5$ and $(5+7)/2 = 6$ L/min.
2. The estimated amount is $1(3.5) + 3(6) = 21.5$ L. The table alone does not give the exact amount.
3. For a concave-down rate graph, the chords lie on or below it, so the estimate is at most the true amount. It is strictly low if the graph rises above a chord anywhere on one of the two panels.

The longer interval contributes three times its estimated rate. Taking an unweighted mean of the three samples loses that information.

Example 2

Write a right-endpoint sum for $\int_2^5 x^2\,dx$ without evaluating it.

1. The total width is 3, so $\Delta x = 3/n$. Start from 2 and move i widths to get $x_i = 2 + 3i/n$.
2. The finite sum is $\sum_{i=1}^{n} \left(2 + 3i/n\right)^2 \left(3/n\right)$. Its limit as n tends to infinity equals the integral.

The expression inside the square is a sample location. The final factor is the width of a rectangle.

> **Common error.** A negative integral of a velocity represents a net displacement in the negative direction. To measure distance traveled, accumulate the absolute value of the velocity.

Explain it aloud

On a table with gaps of 1, 2, and 5 minutes, explain exactly where those three numbers enter a trapezoidal estimate.

LESSON 09 / CED 6.1

Signed area and accumulated change

Interpret the area under a rate graph as a change, including its sign and units.

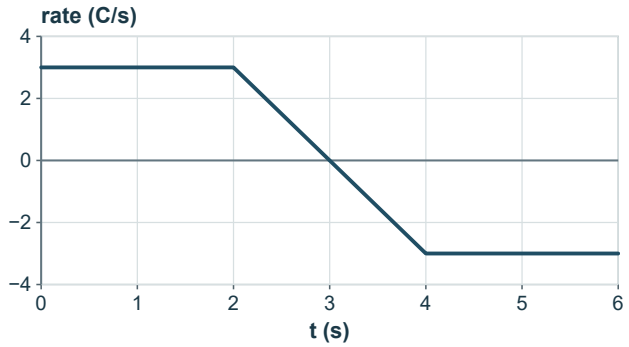
CHA-4.A • OpenStax Calculus Volume 1, §§5.2 and 5.4

Start with a constant rate. A tank gaining 3 liters per minute for 4 minutes gains 12 liters. A rectangle of width 4 minutes and height 3 liters per minute has the same numerical area. A varying rate works similarly: add narrow vertical strips. Contributions above the time axis are positive and contributions below it are negative.

Separate three questions. Signed area gives net change. Initial amount plus signed area gives final amount. Adding the magnitudes of changes gives total variation. For a piecewise-linear rate, rectangles, triangles, and trapezoids often give exact values.

Worked example 1

The graph shows net charge flowing into a battery, in coulombs per second. Find the net charge gained from 0 to 6 seconds.



1. From 0 to 2, the rectangle contributes $2\,(\,3\,) = 6$ coulombs. From 2 to 3, the positive triangle contributes $1\,(\,3\,)\,/2 = 1.5$.
2. From 3 to 4, the triangle contributes -1.5. From 4 to 6, the rectangle contributes -6. The signed sum is 0 coulombs.

Worked example 2

Water leaks out at a constant 1.5 L/min for 8 minutes; a valve then adds water at 4 L/min for 5 minutes. The tank initially holds 50 L. How much water is in it at the end?

1. Use a negative sign for loss and a positive sign for gain. The change is $(-1.5)\,(8) + (4)\,(5) = 8$ L.
2. Add the starting amount: $50 + 8 = 58$ L. The 32 L obtained by adding magnitudes measures the combined volume moved in both directions.

Check your understanding

6.1 C 1 A freezer warms at 2 degrees Celsius per hour for 90 minutes. What is the area under the rate graph in degrees Celsius?

6.1 C 2 A velocity–time graph uses meters per second vertically and seconds horizontally. Over a time interval its signed area has numerical value -6. Supply the physical units and interpret the result.
Explain your reasoning before turning to the check key.

LESSON 09 / CED 6.2

Choosing and interpreting an approximation

Approximate integrals on equal or unequal subintervals and justify possible error direction.

LIM-5.A • OpenStax Calculus Volume 1, §5.1; Volume 2, §3.6 (midpoint and trapezoidal rules)

For each interval multiply its width by one representative height. Left and right sums sample the respective endpoints; the midpoint sum samples the center. A trapezoid uses the mean of the two endpoint heights. Compute each width separately on a nonuniform table. A mean of endpoint heights generally differs from the actual midpoint height.

For an increasing function a left sum is low and a right sum is high; decreasing reverses those conclusions. For a concave-up function chords lie above the graph, so trapezoids overestimate, while the midpoint rule underestimates. Concave down reverses these conclusions. A few increasing samples alone do not establish that a function increases between them.

Worked example 1

At minutes 0, 2, and 5, an increasing flow rate has values 1, 4, and 6 L/min. Bound and estimate the amount with the table.

1. Widths are 2 and 3. The left sum is $2\,(\,1\,)\,+3\,(\,4\,)\,=14$ L and the right sum is $2\,(\,4\,)\,+3\,(\,6\,)\,=26$ L. Monotonicity puts the exact answer between these.
2. The trapezoidal estimate is $2\,(\,1+4\,)\,/2+3\,(\,4+6\,)\,/2=20$ L. An error direction for it would require more information.

Worked example 2

Use two equal midpoint rectangles and two trapezoids for $\int_0^4 x^2\,dx$. Predict which estimate is larger than the integral.

1. Each width is 2. Midpoints 1 and 3 give $2\left(1^2+3^2\right)=20$.
2. Endpoint heights are 0, 4, and 16. Trapezoids give $2\,(\,0+4\,)\,/2+2\,(\,4+16\,)\,/2=24$. The parabola is concave up, so 20 is low and 24 is high.

Check your understanding

6.2 C 1 An increasing function has values $f\,(\,0\,)\,=-6$, $f\,(\,1\,)\,=-4$, and $f\,(\,3\,)\,=-1$. Use the two indicated subintervals to compute a right-endpoint estimate of $\int_0^3 f\,(\,x\,)\,dx$. Is it an upper or lower estimate? Explain even though the heights are negative.

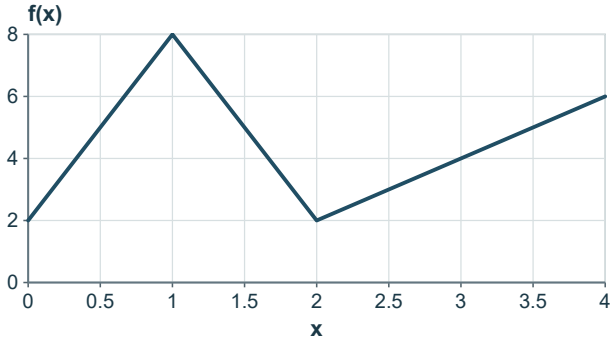
LESSON 09 / CED 6.2 / CONTINUED

Choosing and interpreting an approximation

Approximate integrals on equal or unequal subintervals and justify possible error direction.

LIM-5.A • OpenStax Calculus Volume 1, §5.1; Volume 2, §3.6 (midpoint and trapezoidal rules)

6.2 C 2 First use the three heights at $x = 0, 1, 4$ to find a trapezoidal estimate of the integral on $[0, 4]$. Then use the entire displayed curve, not just those three heights, to compare it with the two secant segments and justify the error direction.



Explain your reasoning before turning to the check key.

LESSON 09 / CED 6.3

Between sums and integrals

Translate in both directions between a limit of sums and an integral and explain what each factor measures.

LIM-5.B / LIM-5.C • OpenStax Calculus Volume 1, §§5.1–5.2

For an interval $[a, b]$ split into n equal pieces, $\Delta x = (b - a)/n$ and right endpoint i is $a + i\Delta x$. The product $f(a + i\Delta x)\,\Delta x$ is one signed rectangle. Adding all n rectangles gives a finite sum. If f is continuous, the limit is $\int_a^b f(x)\,dx$.

Different widths or sample locations are allowed. The largest subinterval width must tend to zero for a general partition. To decode a sum, locate its width, starting point, sample location, and height formula. To construct a sum, name those pieces before writing a sigma.

Worked example 1

Translate $\lim_{n\to\infty} \sum_{i=1}^{n} (5/n)\sqrt{2 + 5i/n}$ into an integral on $[2, 7]$. Interpret one term.

1. Width $5/n$ partitions an interval of length 5. The right endpoint is $x_i = 2 + 5i/n$. The height is $\sqrt{x_i}$.
2. Thus the limit is $\int_2^7 \sqrt{x}\,dx$. Each finite term is the signed area of one width-$5/n$ rectangle.

Worked example 2

Write $\int_{-2}^{1} (1 + x^2)\,dx$ as a left-endpoint limit and identify the first rectangle's height.

1. Width is $3/n$ and the left endpoint of interval i is $-2 + 3(i - 1)/n$. At $i = 1$ the input is -2, so the height is 5.
2. The limit is $\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \left(-2 + 3(i - 1)/n\right)^2\right](3/n)$.

Worked example 3

A continuous rate graph has height $r(t)$ grams per second. On seconds 2 through 6 a sensor totals $\sum_{i=1}^{n} (4/n)\,r(2 + 4i/n)$. Interpret a term and write the meaning of its limit.

1. Each term samples the rate at the right endpoint and multiplies it by $4/n$ seconds. It estimates grams accumulated on one interval.
2. As the widths shrink, continuity of r gives the net accumulated mass $\int_2^6 r(t)\,dt$ in grams.

Worked example 4

For continuous g, represent $\int_2^6 g(x)\,dx$ by a midpoint limit. Compare the first sample with the first right-endpoint sample.

1. The width is $4/n$; midpoint i is $2 + (4i - 2)/n$. The first midpoint is $2 + 2/n$, while the first right endpoint is $2 + 4/n$.
2. The limit is $\lim_{n\to\infty} \sum_{i=1}^{n} (4/n)\,g(2 + (4i - 2)/n)$. Shrinking widths allow either sampling choice to give the same limit.

LESSON 09 / CED 6.3 / CONTINUED

Between sums and integrals

Translate in both directions between a limit of sums and an integral and explain what each factor measures.

LIM-5.B / LIM-5.C • OpenStax Calculus Volume 1, §§5.1–5.2

Check your understanding

6.3 C 1 For $\int_4^{10} \ln x \, dx$ with n equal subintervals, give the width Δx, right-endpoint input x_i, and sampled height $\ln\left(x_i\right)$ separately. Explain the product $\Delta x \ln\left(x_i\right)$. Write the finite right sum and identify its limit as $n \to \infty$, explaining the limiting total.

6.3 C 2 A continuous flow rate is $r(t) = e^t$ L/min on minutes 1 through 3. Interpret one term of $\sum_{i=1}^{n} \left(2/n\right) e^{1+2i/n}$ and give the integral represented by its limit. Why can a rate alone not be added to a volume, and what does shrinking the largest interval width accomplish?

6.3 C 3 Express $\int_{-1}^{3} x^3 \, dx$ as a right-endpoint limit. Identify the width and height of the last rectangle.

6.3 C 4 Give the integral on $[2, 5]$ for $\lim_{n \to \infty} \sum_{i=1}^{n} \left(3/n\right) \ln\left(2 + 3(i-1)/n\right)$. Which end of each interval is sampled?

Explain your reasoning before turning to the check key.

LESSON 09 / CED 6.6

Computing with the properties of integrals

Combine interval addition, orientation, linearity, and signed geometry, including functions with isolated jumps or holes.

FUN-6.A • OpenStax Calculus Volume 1, §5.2

Split an interval wherever the formula or the geometric description changes. Adjacent signed contributions add, even when the integrand is negative. Reversing the bounds changes the sign; identical bounds give zero. Constants factor out and integrals distribute over addition. A constant k contributes $k\,(b-a)$ across $[a,b]$.

One isolated point has zero width. Reassigning its height does not change an integral; a bounded curve with an isolated removable discontinuity can be filled in for integration. At a finite jump integrate each piece separately. These arguments depend on bounded pieces and do not address a vertical asymptote.

Worked example 1

Suppose $\int_1^4 f = 7$ and $\int_4^6 f = -2$ (integrals with respect to x). Find $\int_6^1 (2f(x) - 3)\,dx$.

1. Combine the intervals: $\int_1^6 f(x)\,dx = 7 - 2 = 5$. The constant 3 contributes $3\,(6-1) = 15$.
2. The forward integral is $2\,(5) - 15 = -5$, so the reversed integral is 5.

Worked example 2

On $[-1,1]$, $f(x) = \sqrt{1-x^2}$ except $f(0) = 8$. On $(1,3]$, $f(x) = -2$. Find $\int_{-1}^3 f(x)\,dx$.

1. The first piece has the area of an upper semicircle of radius 1, namely $\pi/2$. The altered height at 0 contributes no area.
2. The jump at 1 creates a second, rectangular piece with signed area $-2\,(3-1) = -4$. The integral is $\pi/2 - 4$.

Check your understanding

6.6 C 1 If $\int_a^b f = 3$ and $\int_a^b g = -2$, find $\int_b^a (f(x) - 4g(x))\,dx$.

6.6 C 2 Let $f(x) = 2$ on $[0,3]$ except $f(1) = 100$. Compute $\int_0^3 f(x)\,dx$.

Explain your reasoning before turning to the check key.

LESSON 10

Accumulation and the fundamental theorem

Read how an accumulation function changes from its rate graph and use antiderivatives to compute net change.

CED 6.4–6.5, 6.7 • OpenStax Calculus Vol. 1, sections 5.2–5.3

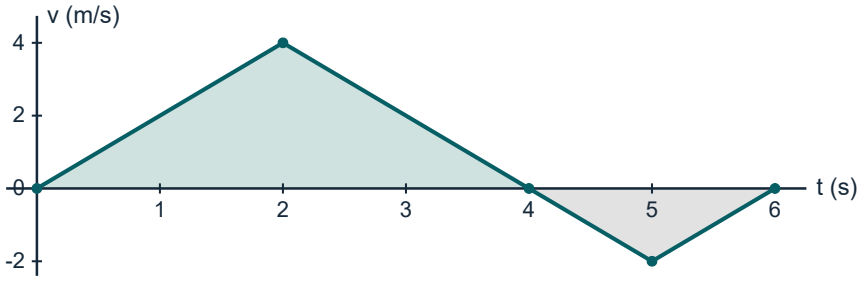
The idea

Suppose r is continuous. The function $A(x) = \int_a^x r(t)\,dt$ records signed accumulation from a fixed starting input a to a moving input x. Its initial value is zero. If the initial quantity was Q_0, the total at x is $Q_0 + A(x)$.

For a small change h, the extra accumulation is $\int_x^{x+h} r(t)\,dt$. Dividing by h gives the average rate over the short interval. As the interval shrinks, this approaches $r(x)$ by continuity. Thus $A'(x) = r(x)$, the differentiation part of the Fundamental Theorem of Calculus.

The sign of r determines whether A rises or falls, and the direction of change of r determines the concavity of A. For an upper bound $g(x)$, the chain rule gives $\frac{d}{dx}\int_a^{g(x)} r(t)\,dt = r\big(g(x)\big)g'(x)$. A moving lower bound contributes the corresponding term with a minus sign.

For continuous r and any antiderivative F of r, the evaluation part of the Fundamental Theorem gives $\int_a^b r(t)\,dt = F(b) - F(a)$. An additive constant cancels in this subtraction. The integral measures a change; adding the initial amount recovers the ending amount.



Velocity is piecewise linear. Its successive vertices are $(0,0)$, $(2,4)$, $(4,0)$, $(5,-2)$, and $(6,0)$.

If the starting position is 3 m, where is the particle after six seconds? How far did it travel?

LESSON 10 / WORKED EXAMPLES

Accumulation and the fundamental theorem

Example 1

A rate graph consists of straight segments joining $(0,2)$, $(2,0)$, $(4,-2)$. Let $A(x) = 3 + \int_0^x r(t)\,dt$. Find $A(2)$, $A(4)$, and the maximum on $[0,4]$.

1. The triangle over $[0,2]$ has signed area 2. The next triangle has signed area -2.
2. Thus $A(2) = 3 + 2 = 5$ and $A(4) = 3 + 2 - 2 = 3$.
3. Since $A' = r$ is positive before 2 and negative after it, the maximum is 5 at $x = 2$.

The zero of the rate locates the turn. The accumulated area and the initial value supply the height of that turn.

Example 2

Let $H(x) = \int_1^{x^2} \cos(t)\,dt$. Find $H'(x)$ and $H(0)$.

1. At the moving upper bound the integrand is $\cos\left(x^2\right)$. The derivative of that bound is $2x$, so $H'(x) = 2x\cos\left(x^2\right)$.
2. For $H(0)$ the bounds are 1 and 0. Evaluating with the antiderivative $\sin t$ gives $H(0) = \sin 0 - \sin 1 = -\sin 1$.

Changing the endpoint requires the chain rule for the derivative; computing a value requires the signed change between two endpoints.

> **Common error.** An accumulation function can be positive and decreasing. Its current value reflects all earlier signed area; its current slope comes from the rate at the present input.

Explain it aloud

Explain how a single graph of a rate gives three different pieces of information: an integral value, the slope of an accumulation graph, and its concavity.

LESSON 10 / CED 6.4

A new function from a moving endpoint

Define an accumulation function, distinguish the integration variable from the input, and differentiate it.

FUN-5.A • OpenStax Calculus Volume 1, §5.3

For a continuous rate r, define $A(x) = \int_a^x r(t)\,dt$. The letter t runs between the bounds; x chooses where accumulation ends. At $x = a$, $A(a) = 0$. From x to $x + h$, the new contribution is $\int_x^{x+h} r(t)\,dt$. Dividing by h gives the average rate on that short interval. As h approaches zero, continuity gives $A'(x) = r(x)$.

For a differentiable moving upper bound $g(x)$, apply the chain rule: the derivative is $r\big(g(x)\big)g'(x)$. A moving lower bound contributes a minus sign. If both move, split at a fixed point or subtract two fixed-start accumulations.

Worked example 1

A reservoir holds 90 kiloliters at hour 2. Its continuous net inflow is $r(t)$ kL/h. Write its amount at hour x, and find its derivative at hour 5 if $r(5) = -3$.

1. Accumulation since hour 2 is $\int_2^x r(t)\,dt$ kL. Therefore $V(x) = 90 + \int_2^x r(t)\,dt$.
2. By the FTC $V'(5) = r(5) = -3$ kL/h. Its amount at that instant decreases at 3 kL/h; finding the amount would need earlier rates.

Worked example 2

For $B(x) = \int_{x^2}^{2x} \sqrt{1 + t^4}\,dt$, find $B'(x)$.

1. Write B as the fixed-start accumulation up to $2x$ minus the fixed-start accumulation up to x^2.
2. The upper endpoint contributes $2\sqrt{1 + 16x^4}$. The lower endpoint contributes $-2x\sqrt{1 + x^8}$. Add the two terms.

Worked example 3

Let $F(x) = \int_1^{\sqrt{x}} \sin t\,dt$ for $x > 0$. Find $F'(x)$ and explain each factor.

1. Let $G(u) = \int_1^u \sin t\,dt$. The integrand is continuous, so the Fundamental Theorem gives $G'(u) = \sin u$.
2. Now $F(x) = G\left(\sqrt{x}\right)$. The upper endpoint moves at rate $1/\left(2\sqrt{x}\right)$, so $F'(x) = \sin\left(\sqrt{x}\right)/\left(2\sqrt{x}\right)$.

Adapted from OpenStax, Calculus Volume 1, §5.3, Example 5.18. Named the outer accumulation function and explained the two derivative factors. Adapted material: CC BY-NC-SA 4.0.

Access for free at https://openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus

LESSON 10 / CED 6.4 / CONTINUED

A new function from a moving endpoint

Define an accumulation function, distinguish the integration variable from the input, and differentiate it.

FUN-5.A • OpenStax Calculus Volume 1, §5.3

Check your understanding

6.4 C 1 At time $t = 3$ min, a bin holds 14 kg. Thereafter it loses material at a continuous positive rate $\ell\,(t)$ kg/min. Write its mass $M\,(x)$ for any later time while material remains. Find $M\,(3)$ and $M'\,(x)$.

6.4 C 2 A tank contains 80 L at hour 1. Its continuous inflow and outflow rates are $i\,(t)$ and $o\,(t)$ L/h. Construct a function for its volume $V\,(x)$ at hour $x \geq 1$, while the model remains physically valid. Find $V'\,(x)$ and explain when the volume decreases.

6.4 C 3 If $H\,(x) = \int_3^x \cos\left(t^2\right)\,dt$, determine $H\,(3)$ and $H'\,(3)$.

6.4 C 4 Differentiate $D\,(x) = \int_{1+x}^4 e^{-t^2}\,dt$.
Explain your reasoning before turning to the check key.

LESSON 10 / CED 6.5

Reading an accumulation graph from its rate

Use the values, signs, and trends of the integrand to determine the height, direction, and concavity of an accumulation function.

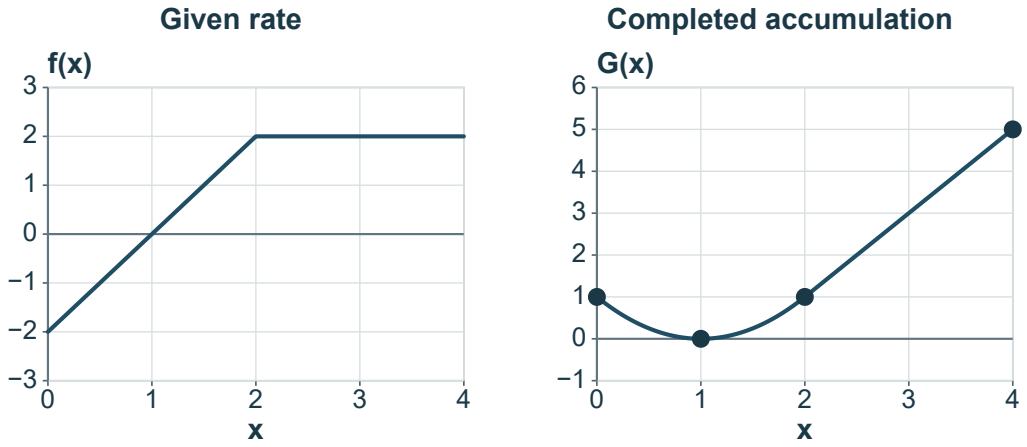
FUN-5.A • OpenStax Calculus Volume 1, §5.3

For continuous f, define $G(x) = k + \int_a^x f(t)\,dt$. Its height comes from past signed areas and the initial value k. Direction comes from the current sign of f, because $G' = f$. Concavity comes from whether f is increasing or decreasing. A negative integrand that rises toward zero makes G decrease and flatten.

To sketch G, mark its starting point, locate where f changes sign, and label where its slope steepens or flattens. Use signed regions to position key points. A zero of f gives a horizontal tangent; test the signs on either side to decide whether G turns.

Worked example 1

The left graph gives f. Define $G(x) = 1 + \int_0^x f(t)\,dt$. Find the minimum and maximum on $[0, 4]$, then use the rate graph to explain the completed accumulation graph at right.



1. Begin at $(0, 1)$. The triangles on $[0, 1]$ and $[1, 2]$ contribute -1 and 1, so $G(1) = 0$ and $G(2) = 1$. The rectangle on $[2, 4]$ adds 4, so $G(4) = 5$.
2. Because $G' = f$, the curve falls until 1, has a horizontal tangent there, and then rises. Increasing f on $(0, 2)$ makes G concave up. Constant $f = 2$ on $(2, 4)$ gives a straight segment of slope 2.
3. The minimum is 0 at $x = 1$ and the maximum is 5 at $x = 4$. As an algebraic check, $G(x) = (x - 1)^2$ on $[0, 2]$ and $G(x) = 2x - 3$ on $[2, 4]$. Both formulas give height 1 and slope 2 at the join.

LESSON 10 / CED 6.5 / CONTINUED

Reading an accumulation graph from its rate

Use the values, signs, and trends of the integrand to determine the height, direction, and concavity of an accumulation function.

FUN-5.A • OpenStax Calculus Volume 1, §5.3

Worked example 2

On $[1, 3]$, a continuous rate f is positive and strictly decreasing, with $\int_1^3 f(t)\,dt = 6$. For $H(x) = \int_3^x f(t)\,dt$, compare $H(1), H(3)$ and describe its shape.

1. Reverse the limits to get $H(1) = -6$, while $H(3) = 0$.
2. The function rises because $H' = f > 0$. Its slopes diminish because f decreases, so it is concave down.

Check your understanding

6.5 C 1 For a continuous function f, let $A(x) = \int_1^x f(t)\,dt$. The signed area from 1 to 4 is -7 and $f(4) = 3$. Is $A(4)$ positive or negative? What is the instantaneous slope?

6.5 C 2 A continuous f touches zero at $x = 2$ and stays negative on both sides. Does $A(x) = \int_0^x f(t)\,dt$ have an extremum there?

Explain your reasoning before turning to the check key.

LESSON 10 / CED 6.7

Evaluating integrals with the fundamental theorem

Use a verified antiderivative to obtain an exact definite integral and interpret its sign.

FUN-6.B • OpenStax Calculus Volume 1, §§5.3–5.4

If f is continuous on $[a, b]$ and $F' = f$, the Fundamental Theorem of Calculus gives $\int_a^b f(x)\,dx = F(b) - F(a)$. Why? The accumulation function $A(x) = \int_a^x f(t)\,dt$ also has derivative f, so A and F differ by a constant. Subtracting their endpoint values cancels that constant.

Write the antiderivative first, then evaluate both endpoints in parentheses. Check by differentiating if the antiderivative came from a guess. A definite integral is a number with context-dependent units. If F is supplied in a table and the integrand is F', the same endpoint subtraction works.

Worked example 1

Evaluate $\int_{-1}^{2} (3x^2 - 4x)\,dx$.

1. The polynomial is continuous. An antiderivative is $F(x) = x^3 - 2x^2$, since $F' = 3x^2 - 4x$.
2. Compute $F(2) = 8 - 8 = 0$ and $F(-1) = -1 - 2 = -3$. Subtract in order: $0 - (-3) = 3$.

Worked example 2

A lab gives $h(1) = 5$, $h(4) = -2$, $h'(1) = 3$, and $h'(4) = 8$, with h' continuous. Compute $\int_1^4 h'(x)\,dx$.

1. Identify an antiderivative of the integrand h': it is h itself.
2. Use $h(4) - h(1) = -2 - 5 = -7$. The endpoint slopes do not enter the evaluation.

Check your understanding

6.7 C 1 Find $\int_0^{\pi/2} \cos x\,dx$ and name the antiderivative you used.

6.7 C 2 If $F(x) = x^2 + 9$ and $G(x) = x^2 - 3$, explain why either works for $\int_1^3 2x\,dx$ and give the value.

Explain your reasoning before turning to the check key.

LESSON 11

Choosing an antiderivative

Recognize a basic antiderivative, expose it by substitution or algebra, and verify the result by differentiation.

CED 6.8–6.10, 6.14 • OpenStax Calculus Vol. 1, sections 4.10, 5.4–5.7

The idea

An indefinite integral represents a family of functions whose derivative is the integrand on the interval of interest. For $p \neq -1$, $\int x^p\,dx = x^{p+1}/(p+1) + C$ wherever the rule applies. For $x \neq 0$, $\int (1/x)\,dx = \ln|x| + C$ on either side of zero. Use an initial value to determine the relevant constant.

Substitution reverses the chain rule. Identify an inner expression $u = g(x)$ and compute $du = g'(x)\,dx$. Rewrite the entire integrand and differential in the new variable. A missing constant factor can be compensated; a leftover variable needs a genuine algebraic substitution.

For a definite integral, replace the bounds a, b by $g(a), g(b)$ when converting to u. Alternatively, find an antiderivative and return to x before using the original bounds. Track the integration variable on every line.

When the numerator degree meets or exceeds the denominator degree, polynomial division may expose a basic rule. Completing the square can reveal $\int du/(1+u^2) = \arctan u + C$. First test whether a denominator's derivative already appears in the numerator. Differentiate your final expression as a check.

> **Think it through.** Given a rational integrand, what would make you choose logarithmic substitution, polynomial division, or completing the square?

LESSON 11 / WORKED EXAMPLES

Choosing an antiderivative

Example 1

Evaluate $\int_0^2 x\sqrt{1+x^2}\,dx$.

1. Choose $u = 1 + x^2$. Then $du = 2x\,dx$ and $x\,dx = du/2$. The new limits are $u = 1$ and $u = 5$.
2. The integral becomes $\frac{1}{2}\int_1^5 \sqrt{u}\,du$. Its value is $\left(5\sqrt{5} - 1\right)/3$.
3. As a check, an antiderivative in the original variable is $\left(1 + x^2\right)^{3/2}/3$. Its derivative is $x\sqrt{1+x^2}$.

A single substitution accounts for the complicated power and the factor multiplying it.

Example 2

Find $\int \frac{x^2 + 2x + 3}{x^2 + 2x + 2}\,dx$.

1. The numerator is the denominator plus 1. Rewrite the integrand as $1 + 1/\left(x^2 + 2x + 2\right)$.
2. Complete the square: $x^2 + 2x + 2 = (x+1)^2 + 1$. With $u = x + 1$, the second term integrates to $\arctan(x+1)$.
3. The full family is $x + \arctan(x+1) + C$. Differentiating gives $1 + 1/\left(1 + (x+1)^2\right)$ as required.

Division exposes the constant term, and completing the square exposes an inverse tangent pattern.

> **Common error.** A logarithm appears when the numerator matches a constant multiple of the denominator's derivative. The antiderivative of $1/\left(1 + x^2\right)$ is $\arctan x$; differentiating $\ln\left(1 + x^2\right)$ would produce an extra factor of $2x$.

LESSON 11 / CED 6.8

Families of antiderivatives

Recover a family of functions from its derivative, use an initial value, and recognize when an integral defines an antiderivative.

FUN-6.C • OpenStax Calculus Volume 1, §4.10 and §5.3

An indefinite integral is a family: $\int f(x)\,dx = F(x) + C$ on an interval where $F' = f$. The power rule runs backward by increasing the exponent by one and dividing by that new exponent, except for exponent -1, which gives $\ln|x|$. Sums and constants integrate term by term. Basic derivative pairs also run backward: for example $\int \cos x\,dx = \sin x + C$, $\int e^x\,dx = e^x + C$, and $\int \sec^2 x\,dx = \tan x + C$. Reversing $(\arcsin x)' = 1/\sqrt{1-x^2}$ gives $\int dx / \sqrt{1 - x^2} = \arcsin x + C$ for $|x| < 1$. The square root distinguishes this from $\int dx / (1 + x^2) = \arctan x + C$.

An initial condition can select a single family member. Verify the derivative and the given value separately. A continuous function always has an antiderivative defined by accumulation. Some, such as e^{-x^2}, have no elementary antiderivative; the definite-integral representation still provides an exact function.

Worked example 1

Find $\int \left(4x^3 - 3 / x + 1 / \sqrt{1 - x^2}\right) dx$ on $-1 < x < 0$.

1. Reverse each derivative separately. The power term gives x^4. On the negative interval a primitive of $-3/x$ is $-3\ln|x|$. The last term matches the derivative of $\arcsin x$, whose formula requires $|x| < 1$.
2. The family is $x^4 - 3\ln|x| + \arcsin x + C$. The specified interval satisfies both domain restrictions. Differentiating each term recovers the original integrand.

Worked example 2

Find the function with derivative $f'(x) = 2e^x + 3\sin x$ and value $f(0) = 4$.

1. Integration gives $f(x) = 2e^x - 3\cos x + C$. Its value at zero is $2 - 3 + C$, so $C = 5$.
2. The selected function is $2e^x - 3\cos x + 5$. Its derivative and its value at zero both check.

Check your understanding

6.8 C 1 Find $\int \left[2 / \sqrt{1 - x^2} + 1\right] dx$ for $|x| < 1$. Verify by differentiation and explain why the reciprocal square root calls for an inverse-trigonometric rule.

6.8 C 2 Give one exact function F with $F'(x) = e^{-x^2}$ and $F(2) = 0$.
Explain your reasoning before turning to the check key.

LESSON 11 / CED 6.9

Undoing a chain rule with substitution

Use substitution for indefinite and definite integrals, converting every factor and both bounds.

FUN-6.D • OpenStax Calculus Volume 1, §§5.5–5.6

A derivative of $F(g(x))$ contains an outer derivative evaluated at $g(x)$ and the factor $g'(x)$. To undo that structure, choose $u = g(x)$ and calculate $du = g'(x)\,dx$. Keep any constant multiplier and replace every occurrence of the original variable. After an indefinite integration, restore the original variable and add C.

For a definite integral change both bounds when you change the variable, even if their order reverses. The transformed bounds are $g(a)$ and $g(b)$. Alternatively, take an indefinite antiderivative, return to x, and then apply the original bounds.

Worked example 1

Find $\int x^2\sqrt{1+x^3}\,dx$ on $x > -1$.

1. Choose $u = 1 + x^3$, so $du = 3x^2\,dx$. The integral becomes $\frac{1}{3}\int u^{1/2}\,du$.
2. Integrate and substitute back: $\frac{2}{9}(1+x^3)^{3/2} + C$. Its derivative recovers $x^2\sqrt{1+x^3}$.

Worked example 2

Evaluate $\int_0^{\pi/2} \sin x/(2+\cos x)\,dx$.

1. Let $u = 2 + \cos x$, so $du = -\sin x\,dx$. The u-bounds are 3 and 2.
2. The integral is $-\int_3^2 du/u = \int_2^3 du/u = \ln(3/2)$. The reversal and the negative differential have both been accounted for.

Check your understanding

6.9 C 1 Find $\int 2x\cos(x^2-3)\,dx$.

6.9 C 2 Using $u = 5 - 2x$, rewrite $\int_0^2 \sqrt{5-2x}\,dx$ as a positive multiple of an integral with increasing bounds. Then evaluate it exactly.

Explain your reasoning before turning to the check key.

LESSON 11 / CED 6.10

Rewriting before integrating

Use polynomial division and completing the square to expose known antiderivatives in both definite and indefinite integrals.

FUN-6.D • OpenStax Calculus Volume 1, §5.7; Volume 2, §3.4 (opening long-division example only)

When a rational expression's numerator has degree at least as large as its denominator, divide. Write the numerator as (quotient)(denominator) + remainder, and divide every term by the original denominator. The simpler pieces may be powers, logarithmic derivatives, or inverse-trigonometric patterns.

For a quadratic denominator complete the square by adding and subtracting the square of half the linear coefficient after making the quadratic monic. For example, $x^2 + 6x + 13 = (x + 3)^2 + 4$. With $a > 0$, the antiderivative of $1/(u^2 + a^2)$ is $(1/a)\arctan(u/a)$. Verify with the chain rule. A numerator proportional to the denominator's derivative calls for a logarithm instead.

Worked example 1

For $x > -2$, find $\int (x^2 + x + 4)/(x + 2)\,dx$.

1. Divide the leading terms: $x^2 / x = x$. Multiply back to get $x(x + 2) = x^2 + 2x$. Subtract the entire product: $(x^2 + x + 4)$ - $(x^2 + 2x) = -x + 4$.
2. Now divide $-x$ by x to get -1. Subtract $-1(x + 2) = -x - 2$ from $-x + 4$, leaving 6. Thus the quotient is $x - 1$ and the remainder is 6: $(x^2 + x + 4)/(x + 2) = x - 1 + 6/(x + 2)$.
3. Integrate to get $x^2/2 - x + 6\ln(x + 2) + C$. The given domain $x > -2$ makes $x + 2$ positive. Differentiating and combining over $x + 2$ recovers the original numerator.

Worked example 2

Evaluate $\int_{-1}^{1} dx/(x^2 + 2x + 5)$.

1. Rewrite $x^2 + 2x + 5 = (x + 1)^2 + 4$. Let $u = (x + 1)/2$, so $dx = 2\,du$ and the new bounds are 0 and 1.
2. The integral becomes $\frac{1}{2}\int_0^1 du/(1 + u^2) = \frac{1}{2}(\arctan 1 - \arctan 0) = \pi/8$.

Check your understanding

6.10 C 1 Find the quotient and remainder in $(x^2 + 5)/(x - 1)$, then give an antiderivative on $x > 1$.

6.10 C 2 Complete the square in $x^2 - 4x + 13$. Give a primitive of its reciprocal and identify the scaling factor.
Explain your reasoning before turning to the check key.

LESSON 11 / CED 6.14

Selecting an integration method

Select and defend an AB integration method before calculating. Revisit basic rules, substitution, and algebraic rewriting.

Skill 1.C • OpenStax Calculus Volume 1, §§5.5–5.7

First inspect the input. A graph or a table can support geometry, estimates, or endpoint subtraction if antiderivative values are given. For a formula, look for a familiar derivative, an inner expression together with its derivative, or algebra that exposes either. A matching denominator derivative suggests a logarithm. A shifted square plus a positive constant can lead to arctangent.

Write a one-line reason for the first move. Then carry it out, keeping original or transformed bounds consistent. Differentiate an indefinite result to test it. For a definite result check sign and size against the integrand on the interval.

Worked example 1

Compare $\int (2x+6)/\left(x^2+6x+10\right)\,dx$ and $\int dx/\left(x^2+6x+10\right)$. Choose a method for each and evaluate.

1. The first numerator is exactly the denominator's derivative. Set $u = x^2+6x+10$ to get $\ln\left(x^2+6x+10\right)+C$; the argument equals $(x+3)^2+1>0$.
2. For the second, the constant numerator leaves the derivative unmatched. Completing the square gives the antiderivative $\arctan(x+3)+C$.

Worked example 2

Choose a first move for $\int_0^1 \left(x^2+2\right)/\left(x^2+1\right)\,dx$ and evaluate.

1. The numerator and denominator have equal degree. Division writes the integrand as $1+1/\left(x^2+1\right)$.
2. Use basic antiderivatives: $[x+\arctan x]_0^1 = 1+\pi/4$. The integrand lies between 1.5 and 2, so its integral over width 1 is consistent with that range.

Check your understanding

6.14 C 1 Which method starts $\int_0^1 xe^{x^2}\,dx$? Carry it through with bounds.

6.14 C 2 A student chooses substitution in $\int \left(x^3-1\right)/(x-1)\,dx$ for $x>1$. Find a shorter route and give the family.

Explain your reasoning before turning to the check key.

LESSON 12

Differential equations and slope fields

Translate a rate law into a differential equation, read its slope field, and build a solution that satisfies an initial condition.

CED 7.1–7.4, 7.6–7.8 • OpenStax Calculus Vol. 2, sections 4.1–4.3 (slope fields only in 4.2); Vol. 1, section 6.8

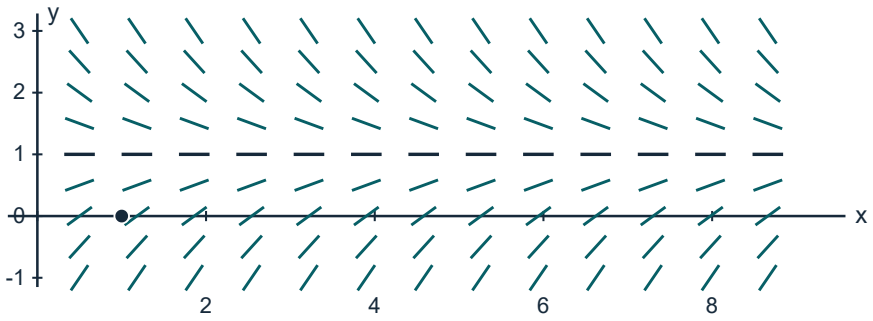
The idea

A differential equation relates an unknown function to one or more derivatives. An initial condition specifies a point on a solution. To verify a proposed solution, differentiate it, substitute into the equation, and check the initial condition separately.

In a slope field for $y' = F(x, y)$, each short segment has the slope computed at that point. A solution curve passing through a point is tangent to its segment. Zero slopes mark horizontal directions. A constant solution requires zero slope all along a horizontal line.

For $y' = g(x)\,h(y)$, first find any constant solutions by solving $h(y) = 0$. On intervals where $h(y) \neq 0$, separate variables as $dy/h(y) = g(x)\,dx$, integrate both sides, and include a constant. Use the initial condition to choose a solution and restrict its domain if the formula breaks down.

A quantity changing at a rate proportional to its present amount satisfies $Q' = kQ$. For $Q(0) = Q_0$, the solution is $Q(t) = Q_0 e^{kt}$. The constant k has inverse-time units. For positive starting amounts, positive k gives growth and negative k gives decay.



The short segments show $dy/dx = 1 - y$. The filled dot marks the initial condition $y(1) = 0$.

Predict the initial direction of the solution, identify an equilibrium, and explain why a solution starting above the equilibrium trends toward it.

LESSON 12 / WORKED EXAMPLES

Differential equations and slope fields

Example 1

For $y' = x - y$, give the slopes at $(0,1)$, $(1,1)$, $(2,1)$. A solution passes through $(1,1)$. Does it turn there?

1. The slopes at those points are $-1, 0, 1$. On graph paper, mark a descending segment, a horizontal segment, and an ascending segment at the three points, respectively.
2. Differentiate the slope equation along the solution: $y'' = 1 - y'$. At $(1,1)$ this gives $y'' = 1$.
3. Since the solution has zero slope and positive second derivative there, it has a local minimum. The diagonal $y = x$ is the set of horizontal segments.

A field reports the slope at a location. To classify a turn on one solution, examine how its slope changes along that solution.

Example 2

A liquid cools in a room held at 20°C. Its temperature satisfies $T' = -0.2\,(T - 20)$ in degrees Celsius per minute, with $T\,(0) = 35$. Find $T\,(t)$.

1. The constant temperature $T = 20$ solves the equation. The initial condition places this liquid above 20. Separate variables: $dT/\,(T - 20) = -0.2\,dt$.
2. Integrating gives $\ln|T - 20| = -0.2t + C$. The initial temperature gives $C = \ln 15$. Since $T - 20 > 0$, $T\,(t) = 20 + 15e^{-0.2t}$.
3. The derivative is $-3e^{-0.2t}$, which equals $-0.2\,(T - 20)$, and $T\,(0) = 35$.

Here the temperature difference from the room decays exponentially. The temperature itself approaches 20 degrees Celsius.

> **Common error.** Dividing by a factor containing y can discard a legitimate solution. For $y' = xy$, record $y = 0$ before dividing by y.

Explain it aloud

How would you distinguish an equilibrium solution from a single horizontal segment in a slope field?

LESSON 12 / CED 7.1

Translate a rate description into an equation

Write a differential equation and initial condition from a description, defining variables and units.

FUN-7.A • OpenStax Calculus, Vol. 2, §4.1

A differential equation relates an unknown function and at least one of its derivatives. Choose the input and name the changing quantity. Translate “per unit time” into a time derivative and “proportional to” into multiplication by a constant. Record a starting amount separately as an initial condition. Check that the units on the right match the derivative and that the predicted sign agrees with the situation.

Worked example 1

A barrel receives 5 L/min and loses 0.12 times its current volume each minute. Initially it contains 18 L. Write the model and find the initial net rate.

1. Let $V(t)$ be volume in liters at t minutes. Inflow minus outflow gives $V' = 5 - 0.12V$, $V(0) = 18$.
2. The constant 0.12 has units per minute. The initial rate is $5 - 0.12(18) = 2.84$ L/min, so volume initially increases.

Worked example 2

A cart experiences acceleration opposite to its velocity and proportional to its speed. Choose east as positive. Write equations for a cart moving east at 6 m/s initially.

1. Let s be position, $v = s'$ signed velocity, and t seconds. With positive constant k in s^{-1}, $v' = -kv$, equivalently $s'' = -ks'$.
2. The initial condition is $v(0) = 6$. The model also gives positive acceleration when velocity is negative, so acceleration always opposes nonzero velocity.

Check your understanding

7.1 C 1 Paint dries at a rate proportional to its wet mass. Write an equation for wet mass M with initial mass 80 g and positive constant k.

7.1 C 2 A bag's volume increases at $0.5t$ L/min, where t is in minutes. Does this give $V = 0.5t$ or $V' = 0.5t$? Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.2

Verify a proposed solution

Test a candidate on an interval and check the initial condition separately.

FUN-7.B • OpenStax Calculus, Vol. 2, §4.1

Differentiate the candidate. Substitute the proposed function on the other side of the equation and check that the two expressions agree throughout the claimed interval. Agreement at one point is insufficient. Some differential equations have infinitely many solutions: varying a free constant can select different functions in one family. An initial condition narrows the candidates, and the formula must be defined throughout the interval claimed.

Worked example 1

Verify that $y = x^2 + Cx$ solves $xy' = y + x^2$ for each real C. Which member goes through $(2, 10)$?

1. $xy' = x(2x + C) = 2x^2 + Cx$. Substituting into the right side gives exactly the same expression.
2. At $x = 2$, $4 + 2C = 10$ implies $C = 3$, so the selected function is $y = x^2 + 3x$.

Worked example 2

For $y' = 2xy$, $y(0) = 1$, test $y = e^{x^2}$ and $y = 1 + x^2$.

1. Both functions take value 1 at zero. For the first, $y' = 2xe^{x^2} = 2xy$ throughout the real line.
2. For the second, $y' = 2x$ but the equation requires $2x\left(1 + x^2\right)$. These disagree at $x = 1$, for instance.

Check your understanding

7.2 C 1 Does $y = 3$ solve $y' = y - 3$? Does it solve the same equation with $y(0) = 4$?

7.2 C 2 For $y' = y^2$ and $y(0) = 1$, verify the candidate $y = 1/(1 - x)$ against both the differential equation and initial condition. State its maximal solution interval containing zero and explain why it cannot include 1.

Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.3

Construct and identify slope fields

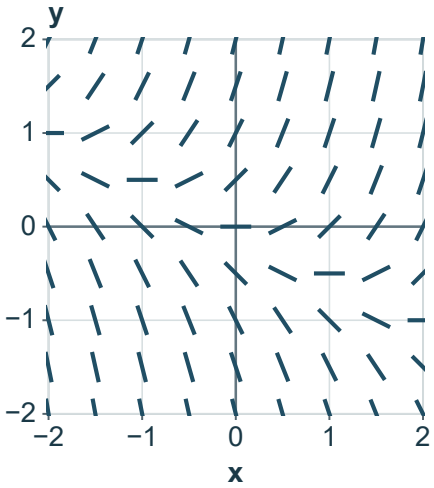
Calculate local slopes on a grid, draw segments, and match a field to its differential equation.

FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, direction fields

An equation $dy/dx = F\left(x, y\right)$ assigns a tangent slope at each point where F is defined. Substitute the coordinates, then draw a short segment centered on the point with that slope. Equal axis scales make visual comparisons of steepness easier. A function of y alone produces identical slopes across each horizontal row. Dependence on x alone produces matching slopes in each vertical column.

Worked example 1

The displayed field corresponds to $y' = x + 2y$. Explain how its middle column was constructed and where else the segments are horizontal.



1. At $(0, -1)$, $(0, 0)$, $(0, 1)$ the slopes are $-2, 0, 2$. Center a descending, flat, and ascending segment at those points.
2. Points with $x + 2y = 0$ have horizontal segments, along $y = -x/2$. Check the field against this diagonal.

LESSON 12 / CED 7.3 / CONTINUED

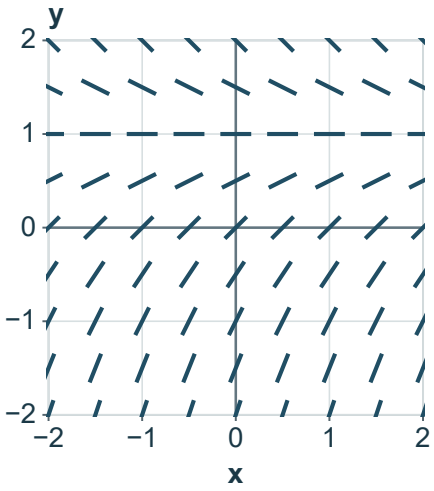
Construct and identify slope fields

Calculate local slopes on a grid, draw segments, and match a field to its differential equation.

FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, direction fields

Worked example 2

Which rule produces the displayed field: $y' = 1 - x$ or $y' = 1 - y$? Compare them at a point off the diagonal.



1. At $(0, 1)$ the printed segment is horizontal. The rule $1 - y$ predicts zero, while $1 - x$ predicts 1.
2. Each row repeats its slopes across x, also consistent with $y' = 1 - y$. The row at $y = 1$ is horizontal.

Check your understanding

7.3 C 1 For $y' = x - y$, compare segments at $(1, -1)$ and $(-1, 1)$.

7.3 C 2 A field has identical slopes down each vertical column but different slopes across columns. Which fits: $y' = 2 - x$ or $y' = 2 - y$?

Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.4

Reason about a solution using a field

Trace a curve through a given point and use the field to explain slopes, turning points, and long-run behavior.

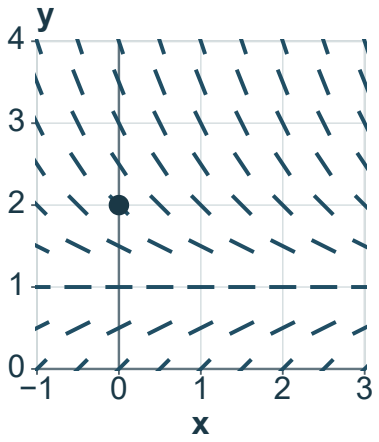
FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, solution curves

A slope field displays local tangent directions. From a specified starting point trace a smooth curve tangent to adjacent segments. Other starting points can select different members of the family. A horizontal segment at one point gives zero derivative there. A full horizontal row of zero slopes can form a constant solution. Use the equation to justify conclusions that extend beyond the printed field.

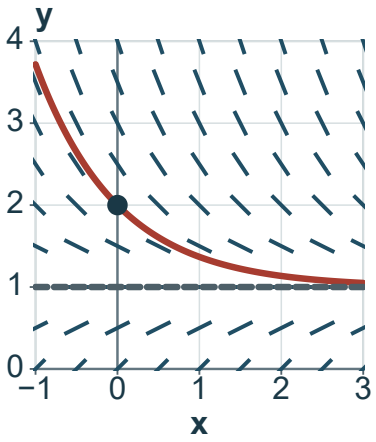
Worked example 1

For the field $y' = 1 - y$, trace the solution through $(0, 2)$ as x increases. Compare it with the solution through $(0, 1)$.

Start with the field



Follow the tangent directions



1. The starting slope is -1. Begin at the dot and draw down and to the right, keeping each short part of the curve tangent to nearby segments.
2. Above $y = 1$ the slopes are negative. As the curve descends toward 1, the slopes become less negative, so the curve bends upward and flattens. The solid trace shows this behavior.
3. Every segment on $y = 1$ is horizontal, so the dashed line is the equilibrium solution through $(0, 1)$. A check of the solid trace is $y = 1 + e^{-x}$, whose derivative equals $1 - y$ and whose value at zero is 2.

LESSON 12 / CED 7.4 / CONTINUED

Reason about a solution using a field

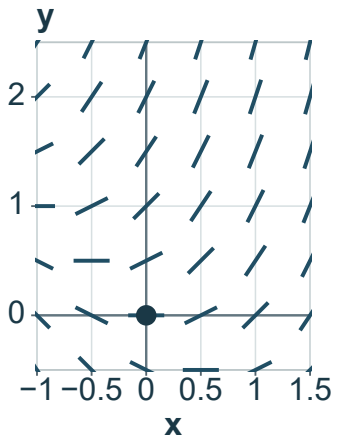
Trace a curve through a given point and use the field to explain slopes, turning points, and long-run behavior.

FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, solution curves

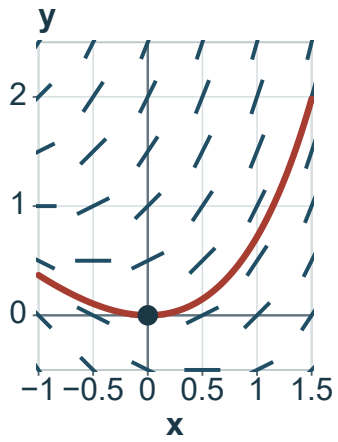
Worked example 2

For the field $y' = x + y$, trace the solution through the origin. Decide whether the origin is a local maximum, local minimum, or neither.

Start with the field



Follow the tangent directions



1. The initial segment is horizontal, so start with a horizontal tangent. A single zero slope does not yet tell whether the solution turns.
2. Above the diagonal $y = -x$ the segments slope upward; below it they slope downward. Near the origin the traced solution falls as x increases from the left and rises to the right. Join the segments smoothly, without drawing the diagonal itself as a solution.
3. The falling-to-rising change gives a local minimum. Differentiating along the solution gives $y'' = 1 + y'$, so $y''(0) = 1 > 0$, confirming the sketch. The plotted trace is $e^x - x - 1$; substituting checks both the equation and the initial point.

LESSON 12 / CED 7.4 / CONTINUED

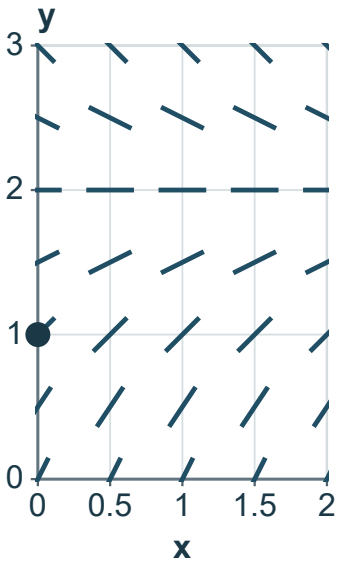
Reason about a solution using a field

Trace a curve through a given point and use the field to explain slopes, turning points, and long-run behavior.

FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, solution curves

Check your understanding

7.4 C 1 On this field for $y' = 2 - y$, sketch the solution through $(0, 1)$ for $0 \le x \le 2$. Estimate $y(1)$ from your sketch and explain whether the curve rises or falls and bends upward or downward. Is $y = 2$ a solution?



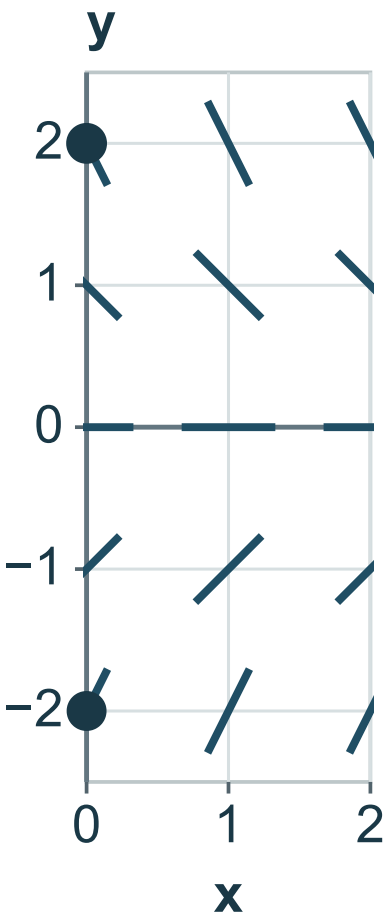
LESSON 12 / CED 7.4 / CONTINUED

Reason about a solution using a field

Trace a curve through a given point and use the field to explain slopes, turning points, and long-run behavior.

FUN-7.C • OpenStax Calculus, Vol. 2, §4.2, solution curves

7.4 C 2 On this field for $y' = -y$, sketch the solutions through $(0, 2)$ and $(0, -2)$ for $0 \le x \le 2$. Estimate both values at $x = 1$. Explain how the field gives the direction and changing steepness of each trace.



Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.6

Separate variables for a general solution

Separate and integrate a differential equation to describe a family of solutions, including any equilibria.

FUN-7.D • OpenStax Calculus, Vol. 2, §4.3

For $y' = g(x)\,h(y)$, check for constant solutions by setting $h(y) = 0$. Where $h(y) \neq 0$, collect the y factors with dy and the x factors with dx, then integrate both sides with one arbitrary constant. Preserve absolute values in logarithms until signs have been addressed. General solutions may be explicit or implicit; check the domain of the original equation.

Worked example 1

Find all solutions to $y' = 3x^2(y - 2)$.

1. The equilibrium is $y = 2$. Otherwise separate $dy/(y - 2) = 3x^2\,dx$.
2. Integrating gives $\ln|y - 2| = x^3 + C$. Exponentiating both sides gives $|y - 2| = e^C e^{x^3}$. The number e^C is positive.
3. On a non-equilibrium solution, $y - 2$ keeps one sign. Thus $y - 2 = Ke^{x^3}$, where $K = e^C$ or $K = -e^C$. This yields $y = 2 + Ke^{x^3}$ for either sign of the nonzero constant K.
4. Setting $K = 0$ separately restores the equilibrium. Check $y' = 3x^2Ke^{x^3} = 3x^2(y - 2)$. Every real K therefore gives a solution defined for all real x.

Worked example 2

Solve $y' = (x + 1)/y$ on intervals where $y \neq 0$. Why can the answer be left implicit?

1. Separate $y\,dy = (x + 1)\,dx$.
2. Integration gives $y^2 = x^2 + 2x + C$, describing both signs of y.
3. Each explicit branch $y = \pm\sqrt{x^2 + 2x + C}$ requires a strictly positive radicand since the original equation excludes $y = 0$.

Check your understanding

7.6 C 1 Give the general solution of $y' = 4x^3$. Why is there a free constant?

7.6 C 2 Find the whole solution family for $y' = x(y + 5)$. Identify the solution that division by $y + 5$ would discard, and explain how the family can include it.

Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.7

Choose the solution through an initial point

Apply an initial condition and check the selected branch, interval, and derivative.

FUN-7.E • OpenStax Calculus, Vol. 2, §§4.1, 4.3

After integrating, substitute the initial point into the family to find its constant. A square root may require selection of a sign. Give the interval containing the initial input where the formula and the original equation both hold. When $y' = f(x)$ with continuous f, the solution satisfying $y(a) = b$ is $y(x) = b + \int_a^x f(t)\,dt$, even if no elementary antiderivative is available.

Worked example 1

Solve $y' = -x/y$ with $y(0) = -3$. Find the largest interval containing zero where the equation holds.

1. Separate $y\,dy = -x\,dx$, yielding $y^2 = -x^2 + C$. The initial point gives $C = 9$.
2. Select $y = -\sqrt{9 - x^2}$. At $x = \pm 3$ the derivative equation is undefined, so the interval is $(-3, 3)$.
3. The derivative is $x/\sqrt{9 - x^2} = -x/y$, and the function takes value -3 at zero.

Worked example 2

A platform's height $H(t)$ has derivative $H'(t) = \cos\left(t^2\right)$ m/s and $H(1) = 4$ meters. Express $H(2)$ exactly.

1. For any time in this model, $H(t) = 4 + \int_1^t \cos\left(u^2\right)\,du$.
2. Thus $H(2) = 4 + \int_1^2 \cos\left(u^2\right)\,du$ meters. The Fundamental Theorem recovers the derivative; at 1 the integral vanishes.

Check your understanding

7.7 C 1 Find y if $y' = 2(y - 4)$ and $y(0) = 4$. Which solution is worth checking first?

7.7 C 2 If f is continuous, $y' = f(x)$ and $y(3) = -2$, construct the solution function $y(x)$. Then express $y(1)$ using bounds 1 and 3, and verify the derivative and initial value.

Explain your reasoning before turning to the check key.

LESSON 12 / CED 7.8

Exponential growth and decay models

Interpret a proportional change law, solve for constants from data, and use the model in context.

FUN-7.F / FUN-7.G • OpenStax Calculus, Vol. 1, §6.8; Vol. 2, §4.3

When the rate of change of a positive quantity Q is proportional to its size, $Q' = kQ$ and the relative rate $Q'/Q = k$ is constant. Its sign distinguishes growth from decay and its units are inverse time. Separation yields $Q(t) = Q_0 e^{kt}$ when $Q(0) = Q_0$. Over a finite interval of length h, the multiplier is e^{kh}: a percentage increase over one day generally differs from the continuous per-day coefficient.

Worked example 1

A drying sample falls from 50 g to 40 g in 3 hours at a rate proportional to its remaining mass. Find its model, interpret k, and predict mass at 6 hours.

1. Write $M' = kM$. The general solution is $M(t) = Ce^{kt}$; positive mass requires $C > 0$. The initial condition $M(0) = 50$ sets $C = 50$. Then $40 = 50e^{3k}$ gives $k = \ln(0.8)/3$ per hour.
2. The relative rate is about -0.0744 per hour. Each three-hour interval leaves 80 percent of the prior mass.
3. The particular model is $M(t) = 50(0.8)^{t/3}$ g, yielding $M(6) = 32$ g.

Worked example 2

A cart's acceleration is proportional to its positive velocity. Initially it travels at 4 m/s, and at 2 s it travels at 6 m/s. Find velocity as a function of time and acceleration at 2 s.

1. Let $v' = kv$. Its general family is $v(t) = Ce^{kt}$. The initial condition gives $C = 4$, and $6 = 4e^{2k}$ gives $k = \ln(1.5)/2$ per second.
2. The particular velocity is $v(t) = 4(1.5)^{t/2}$ m/s. At 2 s, $a(2) = kv(2) = 3\ln(1.5)$ m/s^2. Since $k > 0$ and $v > 0$, acceleration is positive and grows with velocity.

Check your understanding

7.8 C 1 A positive quantity starts at 50 units, increases exactly eight percent in one day, and satisfies $Q' = kQ$. Calculate k, write the model, and explain what happens over each whole day.

7.8 C 2 A sample begins with 120 g and obeys $M' = -0.06M$ for time in hours. Give the general solution and the particular mass function. How rapidly is the mass changing after 2 hours?

7.8 C 3 A balance earns continuously at a rate $B' = 0.04B$ dollars per year. Give the general solution, then choose the member with $B(0) = 2500$. Interpret $B'(0)$.

7.8 C 4 A moving cart slows according to $v' = -0.8v$, with time in seconds. At $t = 2$ its velocity is 5 m/s. Find $v(t)$ and interpret the negative sign in the model.

Explain your reasoning before turning to the check key.

LESSON 13

Net change, average value, and motion

Use an integral to recover an amount, describe a typical value, or measure distance. Choose the integrand by its units and meaning.

CED 8.1–8.3 • OpenStax Calculus Vol. 1, sections 5.2, 5.4

The idea

If $Q'(t) = r(t)$ and the net rate r is continuous on $[a, b]$, the signed area under r over $[a, b]$ gives the change in Q. Add the initial amount to recover the ending amount: $Q(b) = Q(a) + \int_a^b r(t)\,dt$. When two rates describe flow, take inflow minus outflow before integrating.

The average value of a continuous function f over $[a, b]$, with $b > a$, is $\frac{1}{b-a}\int_a^b f(x)\,dx$. This is the height of a rectangle with the same width and signed area. The average value has the same units as f.

For position s, velocity is $v = s'$ and acceleration is $a = v'$. Integrating acceleration gives a change in velocity. Integrating velocity gives displacement. For total distance, integrate speed $|v|$. Split at changes in the sign of velocity if evaluating by hand.

To find the greatest amount stored on a closed interval, compare amounts at the endpoints and at interior critical times. A positive net rate means storage is increasing, even when the inflow rate itself is decreasing.

> **Think it through.** If two trips have the same starting and ending positions and duration, which average must match? Which average can differ? Explain with a sketch.

LESSON 13 / WORKED EXAMPLES

Net change, average value, and motion

Example 1

A runner starts at position 5 m. Her velocity is $v(t) = (t-1)(t-3)$ m/s for $0 \le t \le 4$ s. Find her final position, total distance, and average speed.

1. The signs of v on the three intervals cut by 1 and 3 are positive, negative, positive. An antiderivative is $F(t) = t^3/3 - 2t^2 + 3t$.
2. At times 0, 1, 3, 4, the values of F are $0, 4/3, 0, 4/3$. The signed displacements are $4/3, -4/3, 4/3$ m. Final position is $5 + 4/3 = 19/3$ m.
3. The three travel distances sum to 4 m. Average speed is $4/4 = 1$ m/s. For comparison, average velocity is $(4/3)/4 = 1/3$ m/s.

A sign change reverses the direction of travel. Every travel segment contributes its nonnegative length to the total distance.

Example 2

A tank starts with 20 L. Water enters at $6 + 2\sin(t^2/4)$ L/h and leaves at 4 L/h for $0 \le t \le 3$ h. Use a calculator to find the final amount and the average net rate.

1. Net rate is $r(t) = 2 + 2\sin(t^2/4)$ L/h. In radian mode, evaluate $\int_0^3 r(t)\,dt \approx 9.112951$ L with numerical integration.
2. The final amount is $20 + \int_0^3 r(t)\,dt \approx 29.113$ L. Divide the change by 3 h to get the average net rate, approximately 3.038 L/h.

The initial volume belongs in the final amount. Averaging the net rate uses the change in volume alone.

> **Common error.** A zero velocity is a candidate for a direction change. Check the sign on both sides. Touching zero and keeping the same sign means motion resumes in the same direction.

LESSON 13 / CED 8.1

The average height of a function

Find the constant value that produces the same accumulation over an interval.

CHA-4.B • OpenStax Calculus Volume 1, §5.2 The Definite Integral (average value); §5.3 The Fundamental Theorem of Calculus (mean value theorem for integrals).

For a continuous function f on $[a, b]$ with $a < b$, the rectangle of signed area $\int_a^b f(x)\,dx$ and width $b - a$ has height $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$. The average value has the same units as the function. In contrast, the secant slope measures change per unit of input. Samples collected at unequally spaced times need their time intervals accounted for.

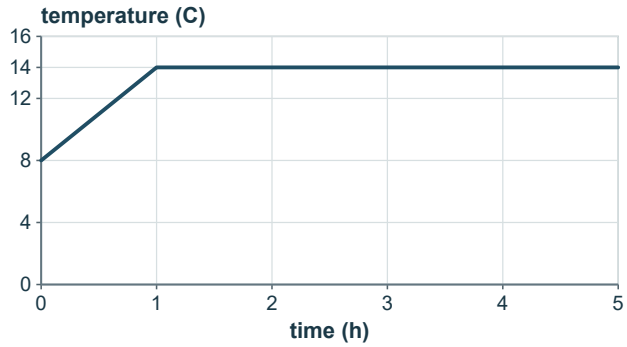
Worked example 1

What is the average value of $f(x) = 4 - x^2$ on $[-1, 2]$? Find a point at which the graph attains that height.

1. The interval width is $2 - (-1) = 3$. The signed area is $\left[4x - x^3/3\right]_{-1}^{2} = 9$.
2. Average height is $9/3 = 3$. Solving $4 - x^2 = 3$ gives $x = -1$ and $x = 1$. Either gives the height, including the interior point $x = 1$.

Worked example 2

The graph shows a sensor reading in degrees Celsius over five hours. Find its average temperature.



1. From hour 0 to hour 1, the area of the trapezoid is $(8 + 14)/2 = 11$ degree-hours. From hour 1 to hour 5, the reading stays at 14, contributing 56 degree-hours.
2. Average temperature is $(11 + 56)/5 = 13.4$ degrees Celsius. The longer interval deserves more weight.

Check your understanding

8.1 C 1 If $\int_2^8 h(t)\,dt = 30$ meter-seconds and h is measured in meters, what is its average height?

8.1 C 2 For $g(x) = x^2$ on $[0, 3]$, is the average of the two endpoint heights the average function value? Explain your reasoning before turning to the check key.

LESSON 13 / CED 8.2

Motion along a line

Use acceleration, signed velocity, and speed to recover changes in velocity, position, and distance.

CHA-4.C • OpenStax Calculus Volume 1, §5.4 Integration Formulas and the Net Change Theorem.

Choose a positive direction first. Given initial velocity at time t_0, use $v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du$, where $a(u)$ is acceleration. Position follows from $s(t) = s(t_0) + \int_{t_0}^{t} v(u)\,du$. Total distance over $[t_0, t_1]$ is $\int_{t_0}^{t_1} |v(t)|\,dt$. Find the zeros of velocity and test signs to determine where to reverse the integrand. A stop can occur without a reversal.

Worked example 1

A cart starts at $s(0) = 3$ meters and has velocity $v(t) = 2t - 4$ m/s for $0 \le t \le 5$. Find its final position and total distance.

1. Velocity changes from negative to positive at $t = 2$. An antiderivative is $F(t) = t^2 - 4t$.
2. Since $F(0) = 0$, $F(2) = -4$, and $F(5) = 5$, signed displacements are -4 and 9 meters. The final position is $3 + 5 = 8$ meters.
3. Total distance is $|-4| + |9| = 13$ meters.

Worked example 2

A particle starts at $v(0) = -1$ m/s and $s(0) = 6$ m. Its acceleration is $a(t) = 6t$ m/s^2. Find $v(2)$ and $s(2)$.

1. Integrate acceleration and use initial velocity: $v(t) = -1 + \int_0^t 6u\,du = 3t^2 - 1$. Thus $v(2) = 11$ m/s.
2. Integrate velocity and use initial position: $s(t) = 6 + \int_0^t (3u^2 - 1)\,du = 6 + t^3 - t$. Hence $s(2) = 12$ m.

Check your understanding

8.2 C 1 A velocity graph encloses 5 square units above the time axis and 8 below it. The units are seconds and m/s. Find displacement and distance.

8.2 C 2 A particle has velocity $v(t) = (t-2)^2$ m/s for $0 \le t \le 3$. Does its stop at 2 seconds require a sign change in the distance integral? Calculate the total distance by integration.

Explain your reasoning before turning to the check key.

LESSON 13 / CED 8.3

Accumulating a changing rate

Interpret an accumulation integral in context and calculate the ending amount or the net change.

CHA-4.D / CHA-4.E • OpenStax Calculus Volume 1, §5.4 Integration Formulas and the Net Change Theorem.

If $Q'(t) = r(t)$ and the net rate r is continuous on $[a, b]$, then $Q(b) - Q(a) = \int_a^b r(t)\,dt$. The integral has units of rate times time. Define $Q(t) = Q(a) + \int_a^t r(u)\,du$ when the question asks how much is present at a particular time. With competing rates, form the signed net rate before integrating. A positive rate increases Q even if that rate is falling.

Worked example 1

A reservoir starts with 50 L. Its inflow is $7 + t$ L/h and outflow is $3 + 2t$ L/h for $0 \le t \le 6$ hours. Find the net change and the largest stored amount.

1. The net rate is $4 - t$. The integral $\int_0^6 (4 - t)\,dt = 6$ L measures the gain in storage during all six hours.
2. Storage increases until $t = 4$ and decreases afterward because $4 - t$ changes sign. At that maximum, $Q(4) = 50 + \int_0^4 (4 - t)\,dt = 58$ L.

Worked example 2

A conveyor deposits gravel at $r(t) = 4 + 1.5 \sin\left(t^2/3\right)$ tonnes per hour on $0 \le t \le 3$. At $t = 0$ the stockpile holds 12 tonnes. Write the stock as a function of time and estimate it at 3 hours to three decimals.

1. The integral accumulates the mass delivered. Including the starting stock gives $Q(t) = 12 + \int_0^t \left[4 + 1.5 \sin\left(u^2/3\right)\right] du$.
2. In radians, numerical integration gives $\int_0^3 r(u)\,du \approx 14.317393$. The final stock is 26.317 tonnes.

Check your understanding

8.3 C 1 A cooling room has temperature H in degrees Celsius. What does $\int_3^7 H'(t)\,dt = -4$ say if t is in hours?

8.3 C 2 A tank contains 40 gallons at noon. Let t be hours after noon. Its inflow is $I(t) = 12 + 2t$ gal/h and outflow is $O(t) = 16 + 4t$ gal/h for $0 \le t \le 1$. Form the signed net rate, calculate the net change by integration, and find the amount at 1 p.m.

8.3 C 3 A bakery's oven cools at a continuous signed temperature rate $T'(t) = -2 + t/2$ °C/min for $0 \le t \le 2$. Evaluate $\int_0^2 T'(t)\,dt$ and interpret the result.

Explain your reasoning before turning to the check key.

LESSON 14

Area between curves

Turn the length of one thin strip into an area integral, choosing the direction that gives the clearest boundaries.

CED 8.4–8.6 • OpenStax Calculus Vol. 1, section 6.1

The idea

A vertical strip extends from the lower graph to the upper graph at a single x. Its height is upper y minus lower y. Multiply by width dx and integrate across the region. A horizontal strip has length right x minus left x and thickness dy.

Sketch both boundaries and label intersection coordinates. For vertical strips, solve for their x-coordinates. For horizontal strips, use their y-coordinates. A region can also end at a specified line or an endpoint of the stated interval.

Check the ordering inside each interval with a convenient test value. Split when a boundary changes or two graphs switch order. At a tangency they may keep the same order. An area is the integral of a nonnegative strip length.

When a calculator supplies intersections, record the equation you solved and store the root at full precision. Use that stored value in each numerical integral. Round only the final area and give square units.

> **Think it through.** Show how a single representative strip determines the variable of integration, the subtraction order, and the units of the answer.

LESSON 14 / WORKED EXAMPLES

Area between curves

Example 1

Find the area between $y = x^3 - x$ and the x-axis on $[-1, 1]$. Both lobes are part of the region.

1. The roots are $-1, 0, 1$. The cubic is above the axis on $(-1, 0)$ and below it on $(0, 1)$.
2. The area is $\int_{-1}^{0} \left(x^3 - x\right)\, dx + \int_{0}^{1} \left(x - x^3\right)\, dx$. The first integral contributes $1/4$ and the second also contributes $1/4$. Total area is $1/2$ square unit.

A single signed integral across both lobes is zero. Each strip in the area calculation has positive length.

Example 2

A region is enclosed by $x = y^2$ and $x = y + 2$. Find its area using horizontal strips.

1. Solve $y^2 = y + 2$ to get $y = -1$ and $y = 2$. At $y = 0$, the line is at $x = 2$ and the parabola is at $x = 0$. The line is the right edge throughout the interior.
2. A horizontal strip has length $y + 2 - y^2$. Its lengths at the endpoints are zero. Area is $\int_{-1}^{2} \left(y + 2 - y^2\right)\, dy = \left[y^2/2 + 2y - y^3/3\right]_{-1}^{2} = 9/2$ square units.

One horizontal strip rule describes the whole region. It also avoids solving the parabola for two separate branches.

> **Common error.** Subtracting the first function mentioned from the second can produce a negative height. Pick a point in each interval and identify the physical top or right edge.

LESSON 14 / CED 8.4

Area from vertical strips

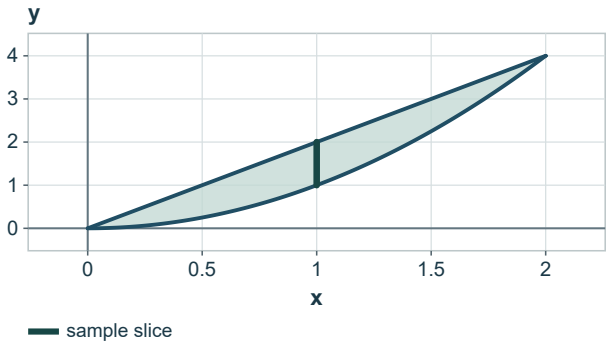
Find a bounded area by integrating the lengths of vertical strips.

CHA-5.A • OpenStax Calculus Volume 1, §6.1 Areas between Curves.

At each x, mark the strip from its lower y-coordinate to its upper y-coordinate. Its length is $y_{\text{top}} - y_{\text{bottom}}$ even if both edges are below the horizontal axis. Find the endpoints of the region from its intersections or stated vertical boundaries. Adding the strip areas gives $\int_a^b \left(y_{\text{top}} - y_{\text{bottom}} \right) dx$.

Worked example 1

Find the area enclosed by $y = 2x$ and $y = x^2$.



1. Solve $2x = x^2$ to get $x = 0, 2$. The line is higher at $x = 1$, so the strip shown has height $2x - x^2$.
2. The area is $\int_0^2 \left(2x - x^2 \right) dx = \left[x^2 - x^3/3 \right]_0^2 = 4/3$ square units.

Worked example 2

The region from $x = 0$ to $x = \pi/2$ lies between $y = -3$ and $y = -3 + \cos x$. Find its area.

1. Cosine is nonnegative here. The upper and lower edges give height $(-3 + \cos x) - (-3) = \cos x$.
2. Its area is $\int_0^{\pi/2} \cos x \, dx = 1$ square unit. A common vertical translation cancels in the strip height.

Check your understanding

8.4 C 1 At $x = 1$, a vertical strip runs from $y = -4$ to $y = -1$. Give its length and its area contribution in a strip of width dx.

8.4 C 2 Find the area enclosed by $y = 4$ and $y = x^2$. Find the intersection bounds, identify the upper curve, and evaluate the area integral with units.

Explain your reasoning before turning to the check key.

LESSON 14 / CED 8.5

Area from horizontal strips

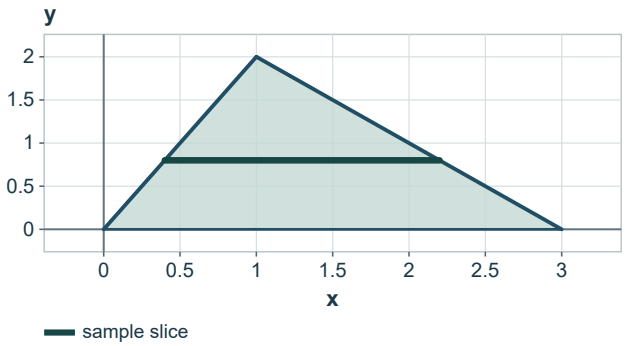
Describe the same plane area with horizontal strips and integrate in the vertical coordinate.

CHA-5.A • OpenStax Calculus Volume 1, §6.1 Areas between Curves.

A horizontal strip at a fixed y extends from the left boundary to the right boundary. Its area contribution is $\left(x_{\text{right}} - x_{\text{left}}\right)\,dy$. Find the top and bottom y-coordinates of the region. Choose the direction that keeps the boundary rules simple. When solving a squared equation for a boundary, use the branch actually present in the region.

Worked example 1

Find the area bounded by $y = 0$, $y = 2x$, and $y = 3 - x$. Which strip direction avoids splitting the integral?



1. Solve for the side boundaries in terms of y: left $x = y/2$ and right $x = 3 - y$. They meet when $y/2 = 3 - y$, giving $y = 2$. The bottom is $y = 0$.
2. One horizontal strip works throughout. Area is $\int_0^2 \left[\left(3 - y\right) - y/2\right] dy = \left[3y - 3y^2/4\right]_0^2 = 3$ square units.
3. With vertical strips, the upper edge changes at $x = 1$. The equivalent calculation is $\int_0^1 2x\,dx + \int_1^3 \left(3 - x\right)\,dx = 1 + 2 = 3$. Horizontal strips avoid this split.

Worked example 2

The region under $y = 4 - x^2$ and above $y = 0$ includes both sides of the parabola. Find its area with respect to y.

1. For $0 \le y \le 4$ the endpoints of the strip are $x = -\sqrt{4 - y}$ and $x = \sqrt{4 - y}$. Its length is $2\sqrt{4 - y}$.
2. Substituting $u = 4 - y$ gives $\int_0^4 2\sqrt{4 - y}\,dy = 2\int_0^4 \sqrt{u}\,du = 32/3$ square units.

Check your understanding

8.5 C 1 For $0 \le y \le 1$, a region extends horizontally from $x = -2 - y^2$ to $x = -1$. Find its area by integrating horizontal strip lengths, including units.

8.5 C 2 A region lies between the positive branch of $y = x^2$ and the y-axis on $1 \le y \le 4$. Identify its right boundary, choose the differential, and calculate its area.

Explain your reasoning before turning to the check key.

LESSON 14 / CED 8.6

Area when curves change order

Partition at crossings to find the sum of geometric areas.

CHA-5.A • OpenStax Calculus Volume 1, §6.1 Areas between Curves.

Solve for every intersection in the interval and test the boundary order between successive intersections. Where the order reverses, reverse the subtraction. The sum equals $\int_a^b |f(x) - g(x)|\,dx$. An intersection where the graphs touch but keep the same order does not require a change of formula. On a calculator, find and store the crossings before evaluating the pieces.

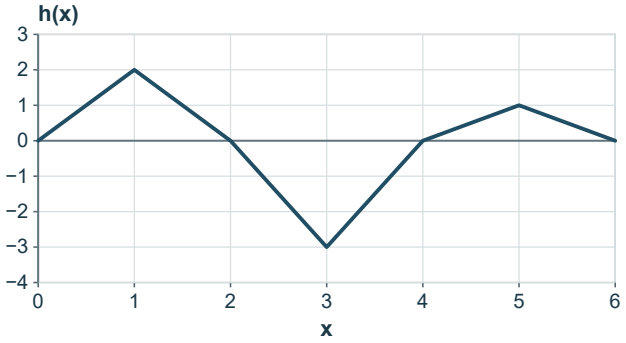
Worked example 1

Find the total area between $f(x) = x^3$ and $g(x) = 4x$ from $x = -2$ to $x = 2$.

1. The difference $x(x-2)(x+2)$ vanishes at $-2, 0, 2$. The cubic lies above the line on $(-2, 0)$ and below it on $(0, 2)$.
2. The two lobe areas agree by symmetry. The total is $2\int_0^2 (4x - x^3)\,dx = 2\left[2x^2 - x^4/4\right]_0^2 = 8$ square units.

Worked example 2

The graph shows the difference $h = f - g$ for two curves. Find the signed integral of the difference and the total area between the curves on $[0, 6]$.



1. The three triangular lobes have base 2 and heights $2, 3, 1$. The signed contributions are $+2, -3, +1$.
2. Their signed sum is 0. Reversing the subtraction on the middle lobe gives total area $2 + 3 + 1 = 6$ square units.

Check your understanding

8.6 C 1 If crossings occur at $a < c < b$ and $f > g$ before c while $g > f$ after c, write the total area.

8.6 C 2 The gap between an upper and lower curve is $(x-1)^2$ on $[0, 2]$. Must the order switch at $x = 1$? Calculate the total area between the curves.

Explain your reasoning before turning to the check key.

LESSON 15

Volumes by slicing

Build a volume from cross-sectional areas and use distances to the axis to set up disk and washer integrals.

CED 8.7–8.12 • OpenStax Calculus Vol. 1, section 6.2

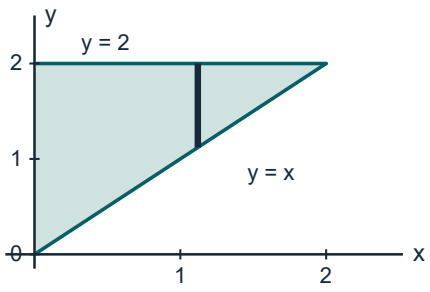
The idea

A slice perpendicular to the x-axis has thickness dx; one perpendicular to the y-axis has thickness dy. If $A(x)$ is the area of a cross section, volume is $\int_a^b A(x)\,dx$. A strip length from the base usually gives one dimension of the slice.

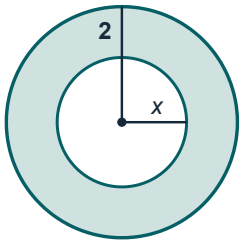
For strip length s, a square with side s has area s^2. A rectangle of height h has area sh. A triangle of base s and altitude h has area $sh/2$. An equilateral triangle with side s has area $\sqrt{3}s^2/4$. A semicircle with diameter s has area $\pi s^2/8$.

In a solid of revolution, disk and washer slices are perpendicular to the axis of rotation. Use vertical strips for a horizontal axis and horizontal strips for a vertical axis. Label the distance from the axis to the far edge R, and the distance to the near edge r. The area is $\pi\left(R^2 - r^2\right)$.

If a strip meets or crosses the axis, rotation fills the center, giving a disk. Its radius is the larger distance to the two endpoints. Split the integral if the slice changes type or the far edge changes. Volumes have cubic units.



Region and a vertical slice



Cross section about the x-axis

The shaded region between $y = x$ and $y = 2$ for $0 \le x \le 2$ rotates about the x-axis. The right panel shows the rotated slice.

Which radius depends on x? Write and evaluate the volume integral.

LESSON 15 / WORKED EXAMPLES

Volumes by slicing

Example 1

A triangular base is bounded by $y = 0$, $x = 0$, and $y = 3 - x$. Compare volumes when cross sections perpendicular to the x-axis are equilateral triangles with one side in the base, or semicircles with their diameters in the base.

1. At position x the base segment has length $s = 3 - x$, for $0 \le x \le 3$. In the triangle the altitude is $\sqrt{3}\,s/2$. In the semicircle the radius is $s/2$.
2. The integral $\int_0^3 (3 - x)^2\,dx = 9$ is common to both calculations. Multiply it by $\sqrt{3}/4$ to get $9\sqrt{3}/4$ cubic units for the triangular slices.
3. Multiply that same integral by $\pi/8$ to get $9\pi/8$ cubic units for the semicircular slices.

The same base segment takes a different role in each area formula. Label that segment before squaring it.

Example 2

Rotate the region enclosed by $y = x^2$ and $y = 2x$ around $y = -1$. Set up the washer integral and evaluate the volume.

1. The curves intersect at $x = 0$ and 2. A vertical strip runs from x^2 up to $2x$. The distances from $y = -1$ are $R = 2x + 1$ and $r = x^2 + 1$.
2. Write $V = \pi \int_0^2 \left[(2x + 1)^2 - \left(x^2 + 1\right)^2 \right] dx$. Expanding the difference gives $-x^4 + 2x^2 + 4x$.
3. The integral is $\pi\left[-x^5/5 + 2x^3/3 + 2x^2\right]_0^2 = 104\pi/15$ cubic units.

Squaring each radius gives the two disk areas. Squaring their difference would lose the dependence on the washer's position relative to the axis.

> **Common error.** A strip width measured across the base can be a side, diameter, or distance between radii. Label its geometric role before choosing an area formula.

Explain it aloud

Sketch a strip rotated around an axis above the region. Point to the edge that produces the outer radius, and explain what changes when the axis passes through the strip.

LESSON 15 / CED 8.7

Square and rectangular cross sections

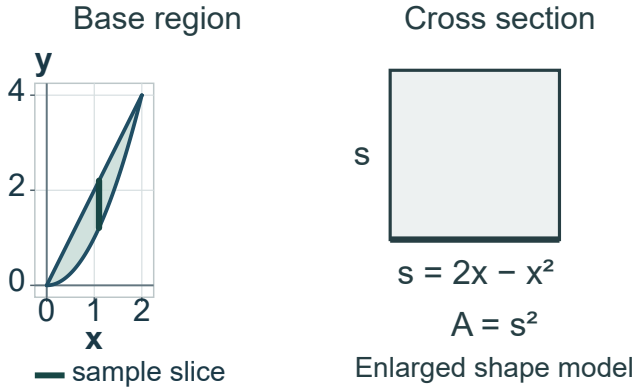
Use the length of a base segment to build a square or rectangular slice, then integrate its area.

CHA-5.B • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing.

Draw a slice perpendicular to the direction in the question. For example, a slice perpendicular to the x-axis uses a vertical segment in the base and has thickness dx. If that segment has length s, a square slice has area s^2. A rectangular slice needs its other dimension h as well, giving $A = sh$. Integrating A times thickness adds thin volumes and produces cubic units.

Worked example 1

The base is enclosed by $y = x^2$ and $y = 2x$. Cross sections perpendicular to the x-axis are squares with a side in the base. Find the solid's volume.



1. The curves cross at $x = 0, 2$. For the shown slice, the side is $s = 2x - x^2$. The entire difference must be squared.
2. Volume is $\int_0^2 \left(2x - x^2\right)^2 dx = \int_0^2 \left(4x^2 - 4x^3 + x^4\right) dx$.
3. Evaluation gives $32/3 - 16 + 32/5 = 16/15$ cubic units.

Worked example 2

A base has horizontal strips from $x = 0$ to $x = 3 - y$ for $0 \le y \le 3$. Each rectangular slice perpendicular to the y-axis has height twice its side in the base. Find the volume.

1. The base side is $3 - y$ and the height is $2\left(3 - y\right)$. Slice area is $2\left(3 - y\right)^2$.
2. Integrating with horizontal thickness gives $V = \int_0^3 2\left(3 - y\right)^2 dy = 18$ cubic units.

Check your understanding

8.7 C 1 At $x = 2$ a base extends from $y = -1$ to $y = 3$. Find the area of a square slice and a rectangular slice of height 5, both using that segment as base.

8.7 C 2 A solid has rectangular cross sections perpendicular to the y-axis. The width is $1 + y$ and height is 3 for $0 \le y \le 2$. Identify the differential, write the volume integral, and evaluate it with units.

Explain your reasoning before turning to the check key.

LESSON 15 / CED 8.8

Triangular and semicircular cross sections

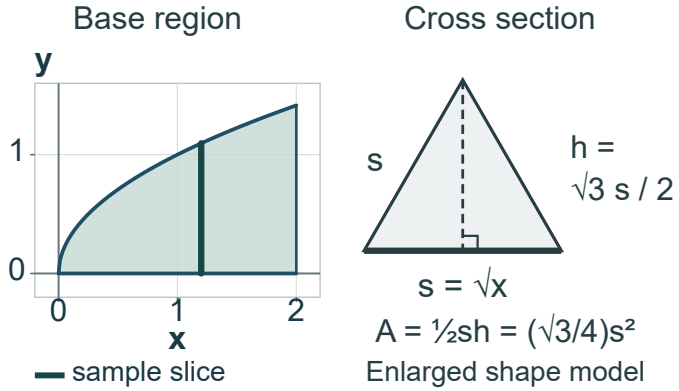
Identify what a base segment measures before writing the cross-sectional area.

CHA-5.B • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing.

For a triangular slice, use one half the base times the perpendicular altitude. Equilateral triangles of side s have altitude $\sqrt{3}\,s/2$. A right isosceles triangle with leg s has area $s^2/2$, while the same triangle with hypotenuse s has area $s^2/4$. A semicircle of diameter s has area $\pi s^2/8$. When an area function is given directly, integrate it with the corresponding thickness.

Worked example 1

The base is $0 \le y \le \sqrt{x}$ for $0 \le x \le 2$. Slices perpendicular to the x-axis are equilateral triangles with one side in the base. Find the volume.



1. The side in the base has length $s = \sqrt{x}$. The altitude of an equilateral triangle is $\sqrt{3}\,s/2$, found by splitting it into two right triangles.
2. The slice area is $\frac{1}{2}s\left(\sqrt{3}\,s/2\right) = \left(\sqrt{3}/4\right)s^2 = \left(\sqrt{3}/4\right)x$. Therefore $V = \left(\sqrt{3}/4\right)\int_0^2 x\,dx = \sqrt{3}/2$ cubic unit.

LESSON 15 / CED 8.8 / CONTINUED

Triangular and semicircular cross sections

Identify what a base segment measures before writing the cross-sectional area.

CHA-5.B • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing.

Worked example 2

For $-2 \le y \le 2$, horizontal segments connect $x = y^2$ to $x = 4$. Those segments are diameters of semicircular slices. Find the volume.

1. Diameter is $d = 4 - y^2$, so the cross-sectional area is $\pi\left(4 - y^2\right)^2/8$. It is zero at either endpoint.
2. Using symmetry, $V = \left(\pi/4\right)\int_0^2 \left(16 - 8y^2 + y^4\right)\,dy = \left(\pi/4\right)\left(32 - 64/3 + 32/5\right) = 64\pi/15$ cubic units.

Check your understanding

8.8 C 1 A right isosceles triangle has hypotenuse 6 cm. What is its area, and why does treating 6 as a leg fail?

8.8 C 2 For $0 \le x \le 4$, a base extends from $y = 0$ to $y = 4 - x$. Cross sections perpendicular to the x-axis are semicircles whose diameters lie in the base. Calculate the volume. How would it change for full circular sections with the same diameters?

Explain your reasoning before turning to the check key.

LESSON 15 / CED 8.9

Disks about a coordinate axis

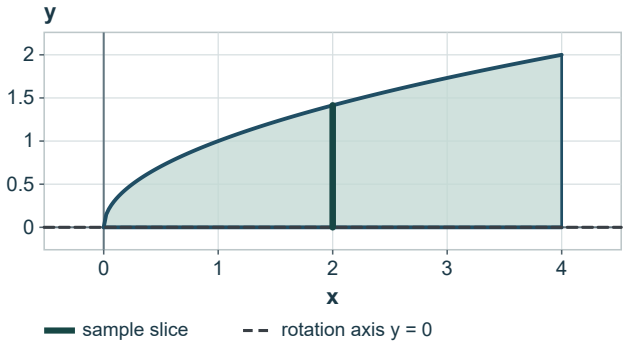
Measure one radius perpendicular to a coordinate axis and add the solid disk slices.

CHA-5.C • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing (disk method).

Rotate a strip perpendicular to the rotation axis. If the strip runs from that axis to a curve, it sweeps a solid disk. The slice area is πR^2 where R is the distance from the axis. A horizontal axis requires vertical strips and dx, while a vertical axis requires horizontal strips and dy. A boundary below or left of the axis can still give a positive radius.

Worked example 1

Rotate the region between $y = 0$ and $y = \sqrt{x}$ for $0 \le x \le 4$ about the x-axis. Find the volume.



1. The shown vertical strip reaches the axis, so it generates a full disk of radius $R = \sqrt{x}$. The cross-sectional area is πx.
2. Volume is $\int_0^4 \pi x \, dx = 8\pi$ cubic units.

Worked example 2

For $-1 \le y \le 1$, rotate the region between $x = y^2 - 1$ and $x = 0$ about the y-axis. Find the volume.

1. The region lies to the left of the axis. The radius is the horizontal distance $0 - \left(y^2 - 1\right) = 1 - y^2$.
2. The disks give $V = \pi \int_{-1}^{1} \left(1 - y^2\right)^2 dy = 2\pi\left(1 - 2/3 + 1/5\right) = 16\pi/15$ cubic units.

Check your understanding

8.9 C 1 A vertical strip from $y = -3$ to $y = 0$ revolves about the x-axis. What shape and area result?

8.9 C 2 A region extends from $x = 0$ to $x = 1 + y$ for $0 \le y \le 2$. Rotate it about the y-axis. Write and evaluate the volume integral, identifying the radius and units.

Explain your reasoning before turning to the check key.

LESSON 15 / CED 8.10

Disks about a shifted axis

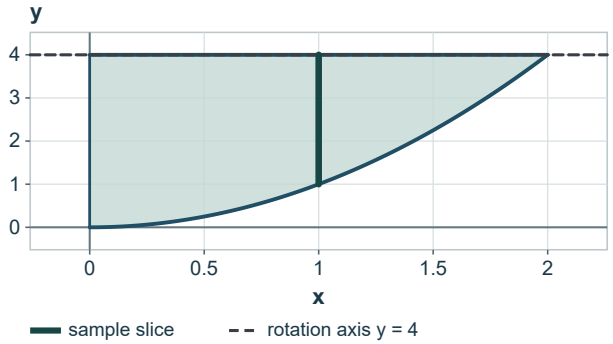
Measure the disk radius from a specified horizontal or vertical line.

CHA-5.C • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing (disk method with other axes).

Draw the line around which the region turns. For rotation about $y = k$, the radius to a point at height $f(x)$ is $|f(x) - k|$. For rotation about $x = h$, use the horizontal distance $|g(y) - h|$. The strip produces a full disk if it touches or crosses the axis. When it crosses the axis, compare both endpoint distances: the farther endpoint determines the radius.

Worked example 1

Rotate the region between $y = x^2$ and $y = 4$ on $0 \le x \le 2$ about $y = 4$. Find its volume.



1. The axis is the top boundary. The distance from it to the lower curve is $R = 4 - x^2$.
2. Squaring gives $V = \pi \int_0^2 (16 - 8x^2 + x^4)\, dx = \pi(32 - 64/3 + 32/5) = 256\pi/15$ cubic units.

Worked example 2

For $0 \le y \le 2$, rotate the region from $x = -2$ to $x = y$ around $x = -2$. Which coordinate gives the thickness, and what is the volume?

1. The vertical axis requires horizontal slices of thickness dy. Their radii are $y - (-2) = y + 2$.
2. Integrate $V = \pi \int_0^2 (y + 2)^2\, dy = \pi\left[(y + 2)^3/3\right]_0^2 = 56\pi/3$ cubic units.

Check your understanding

8.10 C 1 Rotating a strip from $y = 2$ to $y = 5$ about $y = 5$ gives what radius and area?

8.10 C 2 For $0 \le y \le 1$, rotate the region $-2 - y \le x \le 4$ about $x = 1$. Explain whether a hole remains, identify the farther edge, and calculate the volume.

Explain your reasoning before turning to the check key.

LESSON 15 / CED 8.11

Washers about a coordinate axis

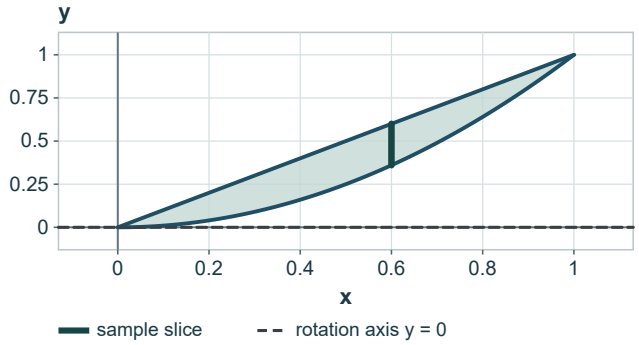
Subtract the area of the central hole from the outer disk before integrating.

CHA-5.C • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing (washer method).

A strip separated from the rotation axis sweeps a washer. Identify its far radius R and near radius r as distances to that same axis. Subtract disk areas to get $\pi R^2 - \pi r^2$. The distance $R - r$ is the ring's thickness and does not determine its area alone. For numerical results, keep full precision until the final volume.

Worked example 1

Rotate the region enclosed by $y = x$ and $y = x^2$ around the x-axis. Find the volume.



1. They meet at $x = 0, 1$. Between these values the line is farther from the axis, so $R = x$ and $r = x^2$.
2. The washer area is $\pi\left(x^2 - x^4\right)$. Integrating gives $V = \pi\left(1/3 - 1/5\right) = 2\pi/15$ cubic unit.

Worked example 2

For $0 \le y \le 1$, rotate $-e^y \le x \le -1$ about the y-axis. Give the volume to three decimal places.

1. The left edge is farther from the axis: $R = e^y$ and $r = 1$. Use horizontal slices.
2. Volume is $\pi \int_0^1 \left(e^{2y} - 1\right)\, dy = \pi\left(e^2 - 3\right)/2 \approx 6.894$ cubic units.

Check your understanding

8.11 C 1 A washer has outer radius 5 cm and inner radius 3 cm. Compute the area and explain why $\pi\left(5 - 3\right)^2$ gives a different quantity.

8.11 C 2 For $0 \le y \le 2$, rotate the region $-4 \le x \le -\left(2 + y\right)$ about the y-axis. Identify both radii and calculate the volume.

Explain your reasoning before turning to the check key.

LESSON 15 / CED 8.12

Washers about a shifted axis

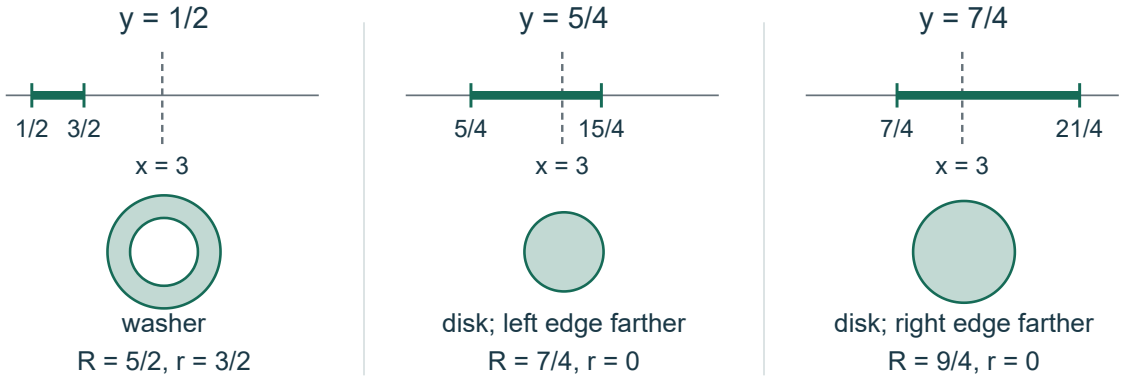
Choose far and near radii from a shifted axis and split the integral when the shape of a slice changes.

CHA-5.C • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing (washer method with other axes).

Mark the axis on the sketch before writing either radius. A horizontal axis above the region puts the lower curve farther away. A vertical axis to the right puts the left curve farther away. For strips that miss the axis, use $\pi\left(R^2 - r^2\right)$. For strips that reach it, set the inner radius to zero and use the farther edge for R. Check where either description changes.

Worked example 1

For $0 \le y \le 2$, the region extends horizontally from $x = y$ to $x = 3y$. Rotate it around $x = 3$. Find the volume, including every point where the slice formula changes.



1. The strip is $[y, 3y]$. The axis first reaches its right end when $3y = 3$, or $y = 1$. For $0 \le y \le 1$, the strip lies left of the axis: $R = 3 - y$ and $r = 3 - 3y$.
2. For $1 \le y \le 2$, the strip crosses the axis, so its rotation has no hole. Compare the endpoint distances $3 - y$ and $3y - 3$. They agree at $y = 3/2$. Thus the left edge sets R on $[1, 3/2]$ and the right edge sets it on $[3/2, 2]$.
3. The volume is $\pi \int_0^1 \left[\left(3 - y\right)^2 - \left(3 - 3y\right)^2 \right] dy + \pi \int_1^{3/2} \left(3 - y\right)^2 dy + \pi \int_{3/2}^2 \left(3y - 3\right)^2 dy$.
4. The three integrals without π are $10/3$, $37/24$, and $21/8$. Therefore $V = \pi\left(80 + 37 + 63\right)/24 = 15\pi/2$ cubic units. At $y = 1$, both slice formulas give area 4π; at $y = 3/2$, both outer radii are $3/2$.

LESSON 15 / CED 8.12 / CONTINUED

Washers about a shifted axis

Choose far and near radii from a shifted axis and split the integral when the shape of a slice changes.

CHA-5.C • OpenStax Calculus Volume 1, §6.2 Determining Volumes by Slicing (washer method with other axes).

Worked example 2

For $0 \le y \le 2$, the region from $x = 1 + y$ to $x = 3$ rotates about $x = 5$. Find the volume and explain the ordering of the radii.

1. The axis is to the right, so the left edge gives the longer distance: $R = 5 - \left(1 + y\right) = 4 - y$. The near radius is $r = 5 - 3 = 2$.
2. The washers give $V = \pi \int_0^2 \left[\left(4 - y\right)^2 - 4\right] dy = \pi\left(24 - 16 + 8/3\right) = 32\pi/3$ cubic units.

Check your understanding

8.12 C 1 For $0 \le x \le 1$, rotate the region $1 \le y \le 3 + x$ about $y = 6$. Find the radii, set up the integral, and calculate the volume.

8.12 C 2 For $0 \le x \le 1$, rotate that same region $1 \le y \le 3 + x$ about $y = 2$. Explain the new slice shape, compare the endpoint distances, and calculate the volume.

UNIT 1 / UNDERSTANDING-CHECK KEY

Limits and continuity

Topics 1.1 through 1.4

1.1 / Change at an instant

1.1 C 1 It is $0/0$. No time elapses in that quotient. Take a limit of valid quotients over intervals with nonzero length.

1.1 C 2 No. The rates from the two sides approach different numbers.

1.2 / Reading and writing limit notation

1.2 C 1 Write $\lim_{u \to 3^+} P(u) = 18$. It describes outputs for inputs just above 3. It supplies neither $P(3)$ nor the left-hand limit.

1.2 C 2 As the input approaches 0 from either side, the output approaches -1. Thus $\lim_{x \to 0^-} g(x) = -1$ and $\lim_{x \to 0^+} g(x) = -1$. The input directions do not specify whether outputs are above or below -1.

1.3 / Estimating limits from graphs

1.3 C 1 No. Values near zero cannot all stay near one height while both separated heights recur indefinitely.

1.3 C 2 Both branches approach 1, so both one-sided limits and the two-sided limit equal 1. The filled point gives $f(-1) = 3$. A low-resolution display may hide the open point at $(-1, 1)$.

1.4 / Estimating limits from tables

1.4 C 1 The left values suggest 2 and the right values suggest 4. Those estimates suggest a jump and no two-sided limit. A finite table alone cannot settle the exact limiting values.

1.4 C 2 No. A constant function of value 1.003 would display 1.00 at that precision and have a different limit.

UNIT 1 / UNDERSTANDING-CHECK KEY

Limits and continuity

Topics 1.5 through 1.8

1.5 / Using limit laws

1.5 C 1 The quotient law cannot be used with a limiting denominator of zero. The sign of q on each side matters.

1.5 C 2 Cosine is continuous at 0. Thus the composite limit is $\cos 0 = 1$.

1.6 / Rewriting expressions before taking limits

1.6 C 1 Since $1 - \cos^2 x = \sin^2 x$, the quotient equals $\sin x$ for nearby nonzero x. Its limit is 0.

1.6 C 2 Only a common multiplicative factor can cancel. At $x = 2$, the original quotient equals $6/4$, while removing the added terms would give $x^2/x = 2$. The unequal results disprove the rewrite.

1.7 / Choosing a limit procedure

1.7 C 1 Follow each branch to $x = 2$ and read the two approached heights. Combine them only if they agree.

1.7 C 2 No. The numerator approaches 6 and shares no factor with the denominator. Check the signs: the quotient tends to $-\infty$ on the left and $+\infty$ on the right.

1.8 / Using the squeeze theorem

1.8 C 1 Both bounds approach 2, so the squeeze theorem gives 2.

1.8 C 2 No. The two bounds approach different numbers. In fact, this oscillation attains both extremes arbitrarily close to zero.

UNIT 1 / UNDERSTANDING-CHECK KEY

Limits and continuity

Topics 1.9 through 1.12

1.9 / Connecting representations of limits

1.9 C 1 $\lim_{x \to 2} f(x) = 5$ and $f(2) = 9$.

1.9 C 2 Yes. Leave the two sampled values on $y = x + 4$ and change only the value at input 1. Each choice produces the same table.

1.10 / Recognizing types of discontinuity

1.10 C 1 No. The common limit is already the point value, so the function is continuous there.

1.10 C 2 Neither. The function stays bounded, and repeated oscillations prevent a common limiting height.

1.11 / Testing continuity at one point

1.11 C 1 The right-hand limit must also be 2. Then the two-sided limit exists and equals the point value.

1.11 C 2 No. The first requirement is a value at 1. A suitable extension can fill the hole.

1.12 / Continuity on intervals

1.12 C 1 On $(2, \infty)$. The argument must be positive, and $\ln 0$ is undefined, excluding 2.

1.12 C 2 It is $\left(-\pi/2, \pi/2\right)$. The adjacent zeros of $\cos x$ make the tangent undefined at each endpoint.

UNIT 1 / UNDERSTANDING-CHECK KEY

Limits and continuity

Topics 1.13 through 1.16

1.13 / Repairing discontinuities

1.13 C 1 No. The left limit is -1 and the right limit is 1. A point assignment leaves those limits unchanged.

1.13 C 2 Solve $3k - 1 = k + 5$, giving $k = 3$. Both sides then equal 8, so set $q(2) = 8$.

1.14 / Infinite limits and vertical asymptotes

1.14 C 1 Simplify any common factors and inspect at least one one-sided limit. A canceled factor can leave a hole with a finite limit.

1.14 C 2 Yes. As $x \to 2^+$ its positive denominator tends to zero and the right-hand limit is $+\infty$. One such side suffices.

1.15 / End behavior and horizontal asymptotes

1.15 C 1 For $x \neq 0$, rewrite $x/(x^2 + 1) = (1/x)/(1 + 1/x^2)$. This tends to 0 at both positive and negative infinity, so both end limits of p are 2 and the horizontal asymptote is $y = 2$. Yet $p(0) = 2$, and the added term changes from negative to positive through 0. A horizontal asymptote describes end behavior and does not forbid a finite-input crossing.

1.15 C 2 Their ratio is $3/(4t) \to 0$. Both rates increase, but r becomes small compared with s.

1.16 / The Intermediate Value Theorem

1.16 C 1 The function is undefined at 0, so it is not continuous on the closed interval. In fact $1/x$ never equals zero.

1.16 C 2 A polynomial is continuous on $[2, 6]$. Since $4 < 10 < 11$, at least one c in $(2, 6)$ satisfies $p(c) = 10$. The endpoints provide no such guarantee for 15, although the graph might reach it.

UNIT 2 / UNDERSTANDING-CHECK KEY

Derivative definitions and rules

Topics 2.1 through 2.4

2.1 / From an average rate to an instantaneous rate

2.1 C 1 $(1.2-1.8)/(5-2)=-0.2$ m/s. The negative sign describes the decrease in height.

2.1 C 2 $\lim_{h\to 0}\left[\left(1/(4+h)\right)-1/4\right]/h$ and $\lim_{x\to 4}\left[\left(1/x\right)-1/4\right]/(x-4)$. Both compare an output with $f(4)$.

2.1 C 3 $[R(2+h)-R(2)]/h=3-h$ for $h\neq 0$. As $h\to 0$ the quotient approaches 3, the instantaneous rate at 2.

2.2 / The derivative function and the tangent line

2.2 C 1 Use $f'(2)=11$ as slope and $(2,4)$ as point. Thus $y-4=11(x-2)$.

2.2 C 2 It is the horizontal line $y=-2$, since every interior tangent to the original line has slope -2.

2.2 C 3 $f'(x)=\lim_{h\to 0}[(1-2(x+h))-(1-2x)]/h=-2$. The graph has constant slope -2.

2.2 C 4 The quotient compares $f(5+h)$ with $f(5)$, so $f'(5)=-3$. The tangent through $(5,7)$ is $y-7=-3(x-5)$.

2.3 / Estimating a derivative from measurements or a graph

2.3 C 1 The estimated slope is $(1-7)/(5-2)=-2$. The chosen coordinates belong to the tangent line.

2.3 C 2 The secant estimate is $(11.4-10.2)/(9.4-8.8)=2$. It averages change over the interval; the samples are not equally spaced about 9. Curvature can make the instantaneous derivative different.

2.4 / Where differentiability fails

2.4 C 1 No. The function lacks a value at the hole and therefore fails continuity, a necessary condition for differentiability.

2.4 C 2 No. Their common finite limit gives $f'(a)=0$; the tangent is horizontal.

UNIT 2 / UNDERSTANDING-CHECK KEY

Derivative definitions and rules

Topics 2.5 through 2.8

2.5 / The power rule

2.5 C 1 It is $-5x^{-6} = -5/x^6$ for $x \neq 0$. The original function already excludes zero.

2.5 C 2 Subtract one from the exponent: $7/2 - 1 = 5/2$. On $x > 0$ the derivative is $\left(7/2\right)x^{5/2}$.

2.6 / Differentiating sums and constant multiples

2.6 C 1 $y' = 3ax^2 - b$. The factors a, b are fixed, and c has zero derivative.

2.6 C 2 They have identical derivatives. If $g = f + C$, then $g' = f' + 0 = f'$.

2.7 / Trig, exponential, and logarithmic derivatives

2.7 C 1 For $x > 0$, the derivative is $3^x \ln 3 + 1/\left(x \ln 2\right)$. The first term has a variable exponent; the second is a logarithm.

2.7 C 2 It is the derivative of $\ln x$ at 2, so its value is $1/2$.

2.7 C 3 $F'\left(x\right) = -\sin x + 1/x$ for $x > 0$. The limit is $F'\left(1\right) = 1 - \sin 1$ in radians.

2.8 / The product rule

2.8 C 1 $2\left(-3\right) + 0\left(10\right) = -6$. The zero factor knocks out one term only.

2.8 C 2 The derivative is $\sin x + x \cos x$, which is $-\pi$ at π.

UNIT 2 / UNDERSTANDING-CHECK KEY

Derivative definitions and rules

Topics 2.9 through 2.10

2.9 / The quotient rule

2.9 C 1 The original function equals $x + 2$ for $x \neq 2$. Thus its derivative equals 1 for every x except 2, where the original function is undefined.

2.9 C 2 $[\,(-1)\,(2) - 4\,(3)\,]/2^2 = -14/4 = -7/2.$

2.10 / Derivatives of the four remaining trig functions

2.10 C 1 Since $\csc\left(\pi/6\right) = 2$ and $\cot\left(\pi/6\right) = \sqrt{3}$, the slope is $-2\sqrt{3}$.

2.10 C 2 $\sec^2\left(3\pi/4\right) = \left(-\sqrt{2}\right)^2 = 2$. The derivative is a positive square at every input where tangent exists.

UNIT 3 / UNDERSTANDING-CHECK KEY

Composite, implicit, and inverse functions

Topics 3.1 through 3.4

3.1 / Rates through composed functions

3.1 C 1 The derivative is $-\sin\left(x^4\right)\cdot 4x^3$; the second factor is the derivative of x^4.

3.1 C 2 It is $12\,(\,0\,)\,=0$. The inside function has zero instantaneous rate there.

3.2 / Differentiating an implicit relation

3.2 C 1 The point gives $2^2\,(\,1\,)^{\,3}=4$. Differentiation gives $2xy^3+3x^2y^2y'=0$. At $(\,2,1\,)$, $4+12y'=0$, so the slope is $-1/3$.

3.2 C 2 The point gives $1\,(\,2\,)+2^2=6$. Differentiation gives $y+xy'+2yy'=0$, so $\left(x+2y\right)y'=-y$. At $(\,1,2\,)$ the coefficient is 5, giving slope $-2/5$.

3.3 / Slope of an inverse function

3.3 C 1 The matching original input is 3. Reciprocal slope gives $1/f'\,(\,3\,)\,=-1/5$.

3.3 C 2 Both $x=2$ and $x=-2$ give 4. Their slopes are 4 and -4, giving different inverse slopes depending on the branch.

3.4 / Differentiating inverse trigonometric functions

3.4 C 1 It is $-2/\sqrt{1-4x^2}$ evaluated at zero, hence -2.

3.4 C 2 They are $2/\sqrt{3}$ and $-2/\sqrt{3}$. The corresponding angles add to $\pi/2$, so their rates are opposite.

UNIT 3 / UNDERSTANDING-CHECK KEY

Composite, implicit, and inverse functions

Topics 3.5 through 3.6

3.5 / Choosing and combining derivative rules

3.5 C 1 The outer fourth power calls for the chain rule. The inner derivative is $1 - \sin x$, giving $4(x + \cos x)^3(1 - \sin x)$.

3.5 C 2 Divide each term by x to obtain $x^2 - 2x + 1$, with derivative $2x - 2$. Retain $x \neq 0$ after simplification.

3.6 / Second and higher derivatives

3.6 C 1 The successive derivatives are $\cos x$, $-\sin x$, $-\cos x$, and $\sin x$.

3.6 C 2 $v(t) = s'(t) = 3t^2 - 12t$ and $a(t) = s''(t) = 6t - 12$. At 3 s, $v = -9$ m/s and $a = 6$ m/s². The particle moves in the negative direction. Velocity is increasing toward zero, so its speed is decreasing.

UNIT 4 / UNDERSTANDING-CHECK KEY

Rates and local models

Topics 4.1 through 4.4

4.1 / Interpreting a derivative in context

4.1 C 1 Yes. At minute 8 the temperature is falling at 0.4 degrees per minute, and it may be above zero.

4.1 C 2 $N'(30) = -4$ tickets per dollar. The price gives the input and the decline gives the sign.

4.2 / Position, velocity, acceleration, and speed

4.2 C 1 Differentiating gives $v(t) = -5 - 2(t-2)$ and $a = -2$. At 2 seconds velocity is -5 m/s and acceleration is -2 m/s^2. Since velocity is negative, speed is $-v$ nearby and increases at 2 m/s^2.

4.2 C 2 $a(t) = v'(t) = 2(t-2)$ m/s^2, so $a(1) = -2$ m/s^2 and $a(3) = 2$ m/s^2. Velocity is positive on both sides of 2, so the stop is not a reversal.

4.3 / Rates beyond straight-line motion

4.3 C 1 $C' = 1.6 + 0.004q$, so near 100 booklets the marginal cost is 2 dollars per additional booklet.

4.3 C 2 The secant rate is $(10 - 14)/(22 - 18) = -1$ watt per degree Celsius. The measurements suggest a decrease locally.

4.4 / Build a related-rates equation

4.4 C 1 $C' = 2\pi r'$ and $A' = 2\pi r r'$. Only the area rate requires the current radius.

4.4 C 2 The constant radius has $r' = 0$. The formula $V' = 2\pi r r' h + \pi r^2 h'$ reduces to $16\pi h'$, so $V' = 4.8\pi$ cubic centimeters per second.

UNIT 4 / UNDERSTANDING-CHECK KEY

Rates and local models

Topics 4.5 through 4.7

4.5 / Solve related-rates problems

4.5 C 1 $V' = 4\pi r^2 r'$, so $8\pi = 16\pi r'$ and $r' = 1/2$ cm/s. The positive rate means the surface moves radially outward at 0.5 cm/s.

4.5 C 2 $A' = (-0.25)(8) + 6(0.5) = 1$ m^2/min, so area increases.

4.6 / Local linearity and tangent estimates

4.6 C 1 Predicted change is $(-3)(-0.1) = 0.3$. The predicted value is 9.3.

4.6 C 2 $L(x) = 2 + 4(x - 1)$ gives $L(1.1) = 2.4$. No error direction follows from those data alone. The functions $2 + 4(x - 1) \pm (x - 1)^2$ share the given value and slope but lie on opposite sides of the tangent.

4.7 / Limits of indeterminate quotients

4.7 C 1 The first limit has form $2/3$, so direct substitution gives $2/3$ and L'Hôpital's rule is ineligible. The second limit has form $0/0$; both functions are differentiable and $2\cos(2x)$ is nonzero near zero. The derivative ratio $3e^{3x}/[2\cos(2x)]$ tends to $3/2$.

4.7 C 2 The form is ∞/∞; the denominator derivative is positive. The derivative ratio $2e^{-0.5x}$ tends to zero.

UNIT 5 / UNDERSTANDING-CHECK KEY

Analyzing functions with derivatives

Topics 5.1 through 5.4

5.1 / A secant slope that a tangent must match

5.1 C 1 The secant slope is zero, so at least one interior tangent is horizontal.

5.1 C 2 No. The theorem gives no guarantee. Another smooth point could have the required slope.

5.2 / Absolute extrema, local extrema, and critical numbers

5.2 C 1 In the first, 1 is outside the domain and is not critical. In the second, 1 is in the domain but the function is discontinuous there, so its derivative does not exist. Thus 1 is a critical number of the second function.

5.2 C 2 The Extreme Value Theorem guarantees that both extreme values are attained on $[2, 5]$. Either can occur at several inputs. If the function is constant, every input attains both.

5.2 C 3 The radicand is nonnegative on the closed bounded interval, so g is continuous there and the EVT guarantees both extrema. The square-root term ranges from 0 to 2. The minimum is 2 at $x = -2$ and $x = 2$; the maximum is 4 at $x = 0$.

5.3 / Intervals of increase and decrease

5.3 C 1 It decreases at a constant rate of four output units per input unit.

5.3 C 2 Yes. For example, $f'(x) = x$ on $(-2, -1)$ rises but remains negative; f decreases there.

5.4 / The first derivative test

5.4 C 1 No. The function continues decreasing through 7.

5.4 C 2 It has a local minimum because the derivative changes from negative to positive.

UNIT 5 / UNDERSTANDING-CHECK KEY

Analyzing functions with derivatives

Topics 5.5 through 5.8

5.5 / Comparing candidates for absolute extrema

5.5 C 1 An endpoint or a different local peak elsewhere in the interval could have a larger value.

5.5 C 2 Both endpoints and the only interior critical number have been checked. Comparing their function values gives the absolute minimum -1 at $x = 1$ and the absolute maximum 8 at $x = 3$.

5.6 / Concavity and changes of concavity

5.6 C 1 No. Its second derivative $12x^2$ is positive on both sides of zero, so concavity does not change.

5.6 C 2 Yes. The derivative can be positive while decreasing. The function rises but its slopes become less steep.

5.7 / The second derivative test

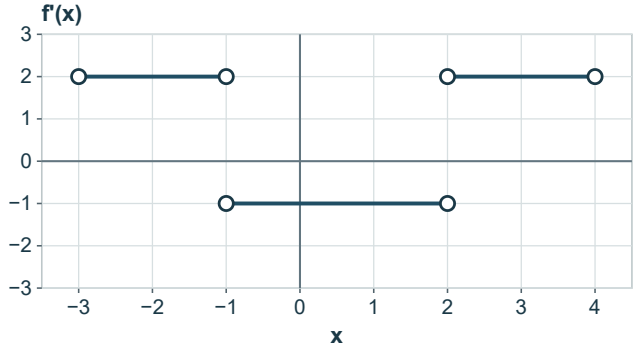
5.7 C 1 No. At this interior point the nonzero slope rules out a differentiable local extremum; the second derivative test requires zero slope.

5.7 C 2 It gives a local minimum. Nothing yet compares it with every point of the domain.

5.8 / Sketching from slopes and curvature

5.8 C 1 Its vertical position. Adding any constant to f preserves the derivative.

5.8 C 2 The segment slopes are $(3 - (-1)) / (-1 - (-3)) = 2$, $(0 - 3) / (2 - (-1)) = -1$, and $(4 - 0) / (4 - 2) = 2$. Thus $f' = 2$ on $(-3, -1)$, $f' = -1$ on $(-1, 2)$, and $f' = 2$ on $(2, 4)$. The unequal one-sided slopes make f' undefined at -1 and 2. Mark open endpoints at each join; the question excludes the outer endpoints.



UNIT 5 / UNDERSTANDING-CHECK KEY

Analyzing functions with derivatives

Topics 5.9 through 5.12

5.9 / Connecting three graphs

5.9 C 1 No. The graph of $f = x^3$ at zero has a horizontal tangent and changes concavity while increasing through zero.

5.9 C 2 The height of the derivative graph gives $f' > 0$, and its negative tangent slope gives $f'' < 0$ at that input. To establish concavity over an interval, check the behavior of f' over that interval.

5.10 / Building an optimization model

5.10 C 1 The constraint is $x^2h = 32$, so $h = 32/x^2$ for $x > 0$. Material area is $A = x^2 + 4xh = x^2 + 128/x$. Then $A' = 2x - 128/x^2 = 2\left(x^3 - 64\right)/x^2$, negative before $x = 4$ and positive after. Thus the global minimum is 48 cm² at $x = 4$ cm and $h = 2$ cm.

5.10 C 2 The other side is $6 - x$, so $A(x) = x(6 - x)$ for $0 < x < 6$. The derivative $6 - 2x$ is positive before 3 and negative after 3. The maximum area is 9 square centimeters, attained by a 3 cm square.

5.11 / Interpreting optimal values in context

5.11 C 1 11 L/min is the minimum rate. The 6 identifies the required setting.

5.11 C 2 No. If the function increases up to 7.4 and decreases afterward across the feasible domain, compare its values at both 7 and 8. A closer input need not give the larger output. Without that monotonic behavior, other feasible integers may do better and must also be considered.

UNIT 5 / UNDERSTANDING-CHECK KEY / CONTINUED

Analyzing functions with derivatives

Topics 5.9 through 5.12

5.12 / Critical points and curvature of implicit curves

5.12 C 1 No. Factoring gives the crossing lines $y = x$ and $y = -x$. They have different slopes at the origin, and neither tangent is vertical.

5.12 C 2 The first differentiation gives $x + yy' = 0$. Differentiating again gives $1 + \left(y'\right)^2 + yy'' = 0$. At $(1, 2)$, $y' = -1/2$ and $y'' = -5/8$, so the branch is concave down nearby. At $(1, -2)$, $y' = 1/2$ and $y'' = 5/8$, so the branch is concave up nearby.

5.12 C 3 Differentiating gives $1 = 2y\,dy/dx$, so $dy/dx = 1/\left(2y\right)$ when $y \neq 0$. At $y = 0$ the curve requires $x = 1$, and $dx/dy = 2y = 0$ gives the vertical tangent $x = 1$ at $(1, 0)$. The point $(0, 0)$ fails the original equation.

5.12 C 4 Differentiation gives $y + xy' = 0$, so $y' = -1$ at $(2, 2)$. A second differentiation gives $2y' + xy'' = 0$, so $y'' = 1$. Both derivatives are continuous on this branch near 2. It decreases and is concave up nearby.

5.12 C 5 $x = 4 - y^2$ gives $dx/dy = -2y$, which vanishes only at $y = 0$. The point is $(4, 0)$; as y passes zero its tangent has zero horizontal change and nonzero vertical change.

5.12 C 6 Differentiating gives $y' = -4x/y$ where $y \neq 0$. Horizontal tangents require $x = 0$ and the original equation gives $y = \pm 2$. At both points the denominator is nonzero.

UNIT 6 / UNDERSTANDING-CHECK KEY

Integration and accumulation

Topics 6.1 through 6.4

6.1 / Signed area and accumulated change

6.1 C 1 Convert time to 1.5 hours. The change is $(2)(1.5) = 3$ degrees Celsius.

6.1 C 2 Multiplying (meters per second) by seconds gives meters. The displacement is -6 m, so the ending position coordinate is 6 m less than the starting coordinate.

6.2 / Choosing and interpreting an approximation

6.2 C 1 The widths are 1 and 2. The right sum is $1(-4) + 2(-1) = -6$. On each subinterval the right height is greatest, so this is an upper estimate of the signed integral. An upper estimate can be negative.

6.2 C 2 The sampled heights are 2, 8, and 6. The estimate is $1(2+8)/2 + 3(8+6)/2 = 26$. The first chord coincides with the graph. The second chord stays above the graph on $(1, 4)$, so its trapezoid overestimates.

6.3 / Between sums and integrals

6.3 C 1 $\Delta x = 6/n$, $x_i = 4 + 6i/n$, and the height is $\ln\left(4 + 6i/n\right)$. The product $\left(6/n\right)\ln\left(4 + 6i/n\right)$ is one signed rectangle area. Adding gives $S_n = \sum_{i=1}^{n}\left(6/n\right)\ln\left(4 + 6i/n\right)$. Since $\ln x$ is continuous on $[4, 10]$ and widths shrink to zero, $\lim_{n\to\infty} S_n = \int_4^{10} \ln x\, dx$, the exact area under this positive curve. Multiplying width by input x_i would use the wrong height.

6.3 C 2 Each term is a sampled rate in L/min times a width $2/n$ minutes, giving an estimated volume in liters. A rate by itself has different units from volume. As the widths approach zero, continuity makes the sum approach the delivered volume $\int_1^3 e^t\, dt$.

6.3 C 3 The limit is $\lim_{n\to\infty}\sum_{i=1}^{n}\left(4/n\right)\left(-1 + 4i/n\right)^3$. The last rectangle has width $4/n$ and height $3^3 = 27$.

6.3 C 4 The width is $3/n$ and $2 + 3(i-1)/n$ is the left endpoint. The integral is $\int_2^5 \ln x\, dx$.

6.4 / A new function from a moving endpoint

6.4 C 1 $M(x) = 14 - \int_3^x \ell(t)\, dt$. The integral measures material lost since minute 3. Thus $M(3) = 14$ kg and $M'(x) = -\ell(x)$ kg/min. The minus sign represents the loss.

6.4 C 2 $V(x) = 80 + \int_1^x [i(t) - o(t)]\, dt$. The initial volume is added to net accumulation. The FTC gives $V'(x) = i(x) - o(x)$ L/h, which is negative when outflow exceeds inflow.

6.4 C 3 The equal bounds give $H(3) = 0$. The FTC gives $H'(3) = \cos 9$.

6.4 C 4 The moving lower bound contributes a minus sign, so $D'(x) = -e^{-(1+x)^2}$.

UNIT 6 / UNDERSTANDING-CHECK KEY

Integration and accumulation

Topics 6.5 through 6.8

6.5 / Reading an accumulation graph from its rate

6.5 C 1 The value is -7 and the slope is $A'(4) = 3$. A negative accumulated value can have positive slope.

6.5 C 2 No. $A'(2) = 0$, but the derivative stays negative on both sides, so A continues decreasing.

6.6 / Computing with the properties of integrals

6.6 C 1 The forward integral is $3 - 4(-2) = 11$. Reversing the bounds gives -11.

6.6 C 2 The isolated point changes no area. The integral is $2(3) = 6$.

6.7 / Evaluating integrals with the fundamental theorem

6.7 C 1 Use $\sin x$. The endpoints give $\sin(\pi/2) - \sin 0 = 1$.

6.7 C 2 Both derivatives equal $2x$. Each endpoint difference is $3^2 - 1^2 = 8$, because its additive constant cancels.

6.8 / Families of antiderivatives

6.8 C 1 The family is $2\arcsin x + x + C$. Differentiating gives $2 / \sqrt{1 - x^2} + 1$. The whole expression $1 - x^2$ is inside the square root; it is not a single power of x, so applying the power rule directly to x would not match the integrand.

6.8 C 2 Take $F(x) = \int_2^x e^{-t^2}\,dt$. Continuity gives the required derivative, and the matching bounds give the initial value.

UNIT 6 / UNDERSTANDING-CHECK KEY

Integration and accumulation

Topics 6.9 through 6.14

6.9 / Undoing a chain rule with substitution

6.9 C 1 Let $u = x^2 - 3$, so $du = 2x\,dx$. The integral is $\sin u + C = \sin\left(x^2 - 3\right) + C$.

6.9 C 2 The new bounds are 5 and 1, and $dx = -du/2$. Combining signs gives $\frac{1}{2}\int_1^5 \sqrt{u}\,du = \frac{1}{3}\left[u^{3/2}\right]_1^5 = \left(5\sqrt{5} - 1\right)/3$.

6.10 / Rewriting before integrating

6.10 C 1 The quotient is $x + 1$ with remainder 6, since $(x - 1)(x + 1) + 6 = x^2 + 5$. Integrating gives $x^2/2 + x + 6\ln(x - 1) + C$.

6.10 C 2 The square form is $(x - 2)^2 + 9$. A primitive is $\frac{1}{3}\arctan\left((x - 2)/3\right)$. The outside factor $1/3$ along with the inside scale produces the required denominator.

6.14 / Selecting an integration method

6.14 C 1 The inner derivative is $2x$, twice the factor present. Use $u = x^2$ and $x\,dx = du/2$. Bounds remain 0 and 1, so the integral is $(e - 1)/2$.

6.14 C 2 Factor $x^3 - 1 = (x - 1)\left(x^2 + x + 1\right)$. Integrate the polynomial to obtain $x^3/3 + x^2/2 + x + C$.

UNIT 7 / UNDERSTANDING-CHECK KEY

Differential equations

Topics 7.1 through 7.4

7.1 / Translate a rate description into an equation

7.1 C 1 $M' = -kM$ and $M(0) = 80$. The minus sign represents loss of wet mass.

7.1 C 2 $V' = 0.5t$. The formula describes the rate. A starting volume would be needed to specify V.

7.2 / Verify a proposed solution

7.2 C 1 It solves the equation because both sides are zero. Its initial value is 3, so it fails the specified initial-value problem.

7.2 C 2 Differentiating gives $y' = 1/(1-x)^2 = y^2$ at every input where the formula is defined. Also $y(0) = 1$. Its maximal interval containing zero is $(-\infty, 1)$, because the function is undefined at 1.

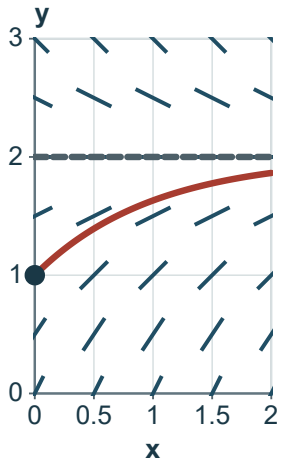
7.3 / Construct and identify slope fields

7.3 C 1 The slopes are 2 and -2. They have equal steepness with opposite signs.

7.3 C 2 $2 - x$ depends only on the column coordinate x; $2 - y$ would repeat slopes across rows.

7.4 / Reason about a solution using a field

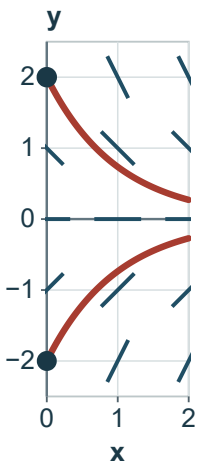
7.4 C 1 The curve rises from $(0, 1)$ and becomes less steep, so it is concave down. A reasonable sketch gives $y(1)$ about 1.6 (accept roughly 1.5 to 1.75 with a compatible trace). The exact check is $y = 2 - e^{-x}$ and $y(1) \approx 1.632$. Every field segment along $y = 2$ is horizontal, so that line is an equilibrium solution.



Differential equations

Topics 7.1 through 7.4

7.4 C 2 The positive branch falls and bends upward; the negative branch rises and bends downward. Both flatten toward $y = 0$. Reasonable estimates at $x = 1$ are about 0.7 and -0.7 (accept roughly 0.6 to 0.9 in magnitude with compatible traces). The exact checks are $y = \pm 2e^{-x}$, giving approximately ± 0.736.



UNIT 7 / UNDERSTANDING-CHECK KEY

Differential equations

Topics 7.6 through 7.8

7.6 / Separate variables for a general solution

7.6 C 1 $y = x^4 + C$. Each vertical shift has derivative $4x^3$.

7.6 C 2 First check $y = -5$. For other solutions $\int dy/(y+5) = \int x\,dx$, so $\ln|y+5| = x^2/2 + C$. Thus $y = -5 + Ke^{x^2/2}$. Any nonzero real K gives a nonconstant solution and $K = 0$ restores the equilibrium.

7.7 / Choose the solution through an initial point

7.7 C 1 The constant function $y = 4$ satisfies both conditions. The family $y = 4 + Ce^{2x}$ selects $C = 0$.

7.7 C 2 $y(x) = -2 + \int_3^x f(t)\,dt$. The Fundamental Theorem gives $y'(x) = f(x)$, and the integral is zero at $x = 3$. Reversing the bounds gives $y(1) = -2 - \int_1^3 f(t)\,dt$.

7.8 / Exponential growth and decay models

7.8 C 1 From $Q(t) = 50e^{kt}$, the one-day multiplier satisfies $e^k = 1.08$, giving $k = \ln(1.08)$ per day. Thus $Q(t) = 50(1.08)^t$. The instantaneous relative rate is k and the one-day increase is eight percent of that day's starting amount.

7.8 C 2 The general solution is $M = Ce^{-0.06t}$. Initial mass sets $C = 120$. At $t = 2$, $M' = -7.2e^{-0.12}$ g/h. Its negative sign means mass is falling at about 6.386 g/h at that instant.

7.8 C 3 The general solution is $B(t) = Ce^{0.04t}$. The initial deposit sets $C = 2500$. Its initial growth rate is $0.04(2500) = 100$ dollars per year.

7.8 C 4 Use the observed time as the anchor: $v(t) = 5e^{-0.8(t-2)}$ m/s. The general family is $Ce^{-0.8t}$. While velocity is positive the acceleration is negative; its magnitude decreases in proportion to the speed.

UNIT 8 / UNDERSTANDING-CHECK KEY

Applications of integration

Topics 8.1 through 8.4

8.1 / The average height of a function

8.1 C 1 Divide by the six-second duration: $30/6 = 5$ meters.

8.1 C 2 Endpoint averaging gives $(0+9)/2 = 4.5$. Integrating gives $\frac{1}{3}\int_0^3 x^2\,dx = 3$. The secant line lies above the curve between the endpoints.

8.2 / Motion along a line

8.2 C 1 Displacement is $5 - 8 = -3$ m. Distance is $5 + 8 = 13$ m.

8.2 C 2 No. Velocity is nonnegative on both sides of 2, so $|v| = v$ throughout. Distance is $\int_0^3 (t-2)^2\,dt = \left[(t-2)^3/3\right]_0^3 = 3$ m.

8.3 / Accumulating a changing rate

8.3 C 1 Between hours 3 and 7, the temperature ends 4 degrees Celsius lower than it began. This does not specify its temperature at either endpoint.

8.3 C 2 The net rate is $I(t) - O(t) = -4 - 2t$ gal/h. The net change is $\int_0^1 (-4-2t)\,dt = \left[-4t - t^2\right]_0^1 = -5$ gallons. Thus the tank contains $40 - 5 = 35$ gallons at 1 p.m.

8.3 C 3 $\left[-2t + t^2/4\right]_0^2 = -3$ °C. The ending oven temperature is 3 °C below its starting temperature. The integral gives the change, so an initial reading is required to give the ending temperature.

8.4 / Area from vertical strips

8.4 C 1 The length is $(-1) - (-4) = 3$, so the area contribution is $3\,dx$.

8.4 C 2 The intersections satisfy $4 = x^2$, so $x = -2, 2$. The line is above the parabola between them. Area is $\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - x^3/3\right]_{-2}^{2} = 32/3$ square units.

UNIT 8 / UNDERSTANDING-CHECK KEY

Applications of integration

Topics 8.5 through 8.8

8.5 / Area from horizontal strips

8.5 C 1 Right minus left is $-1-\left(-2-y^2\right)=1+y^2$. Thus $A=\int_0^1\left(1+y^2\right)dy=\left[y+y^3/3\right]_0^1=4/3$ square units. Being left of the vertical axis does not make a strip length negative.

8.5 C 2 The right edge is $x=\sqrt{y}$ and the left edge is 0. Horizontal slices give $A=\int_1^4\sqrt{y}\,dy=\left[2y^{3/2}/3\right]_1^4=14/3$ square units.

8.6 / Area when curves change order

8.6 C 1 Use $\int_a^c\left(f-g\right)dx+\int_c^b\left(g-f\right)dx$, equivalently $\int_a^b\left|f-g\right|dx$.

8.6 C 2 No. The gap is zero at 1 but is nonnegative on both sides. One integral gives $A=\int_0^2\left(x-1\right)^2dx=\left[\left(x-1\right)^3/3\right]_0^2=2/3$ square unit.

8.7 / Square and rectangular cross sections

8.7 C 1 The side is $3-\left(-1\right)=4$. The areas are $4^2=16$ and $4\left(5\right)=20$ square units.

8.7 C 2 The cross-sectional area is $3\left(1+y\right)$ and thickness is dy. Thus $V=\int_0^2 3\left(1+y\right)dy=\left[3y+3y^2/2\right]_0^2=12$ cubic units.

8.8 / Triangular and semicircular cross sections

8.8 C 1 Each leg is $6/\sqrt{2}$. The area is $\frac{1}{2}\left(6/\sqrt{2}\right)^2=9$ cm^2. Using 6 as a leg would give 18 cm^2.

8.8 C 2 The diameter is $4-x$, so slice area is $\pi\left(4-x\right)^2/8$. Volume is $\left(\pi/8\right)\int_0^4\left(4-x\right)^2dx=\left(\pi/8\right)\left(64/3\right)=8\pi/3$ cubic units. Full circles double each slice area, giving $16\pi/3$ cubic units.

UNIT 8 / UNDERSTANDING-CHECK KEY

Applications of integration

Topics 8.9 through 8.12

8.9 / Disks about a coordinate axis

8.9 C 1 The strip sweeps a disk of radius 3 and area 9π square units.

8.9 C 2 Each horizontal segment touches the axis, so the disk radius is $1+y$. Thus $V=\pi\int_0^2 (1+y)^2\,dy=\pi\Big[(1+y)^3/3\Big]_0^2=26\pi/3$ cubic units.

8.10 / Disks about a shifted axis

8.10 C 1 The radius is the distance $5-2=3$, and the disk area is 9π.

8.10 C 2 Each strip crosses $x=1$, so it sweeps a disk. The left and right distances are $3+y$ and 3. The left edge is at least as far, so $R=3+y$. Volume is $\pi\int_0^1 (3+y)^2\,dy=\pi\Big[(3+y)^3/3\Big]_0^1=37\pi/3$ cubic units.

8.11 / Washers about a coordinate axis

8.11 C 1 Washer area is $\pi(25-9)=16\pi$ cm^2. The expression 4π cm^2 is the area of a full disk of radius 2 cm.

8.11 C 2 The left edge at $x=-4$ is farther from the axis, so $R=4$. The nearer edge gives $r=2+y$. Thus $V=\pi\int_0^2\Big[16-(2+y)^2\Big]\,dy=\pi\big[12y-2y^2-y^3/3\big]_0^2=40\pi/3$ cubic units.

8.12 / Washers about a shifted axis

8.12 C 1 The region lies below the axis. The lower edge gives $R=6-1=5$; the upper edge gives $r=6-(3+x)=3-x$. Thus $V=\pi\int_0^1\Big[25-(3-x)^2\Big]\,dx=\pi\big[16x+3x^2-x^3/3\big]_0^1=56\pi/3$ cubic units.

8.12 C 2 Each strip crosses $y=2$ and sweeps a disk. The lower endpoint is 1 unit away; the upper endpoint is $1+x$ units away. Thus $R=1+x$ and $V=\pi\int_0^1 (1+x)^2\,dx=7\pi/3$ cubic units.

FORMULA REFERENCE / 1

Radians and trigonometry

Learn the standard angles from 0 through $\pi/2$ in order. Sketch a unit-circle point when an angle, sign, or coordinate is uncertain.

How do you convert degrees to radians? How is arc length related to angle?

Multiply degrees by $\frac{\pi}{180}$. An angle of θ radians cuts off arc length $s = r\theta$ on a circle of radius r (use positive θ for length).

On the unit circle, what are the coordinates of the point at angle t? How do you find signs outside the first quadrant?

The point is $(\cos t, \sin t)$. The horizontal coordinate gives the sign of cosine. The vertical coordinate gives the sign of sine.

Give the sines of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ in order.

$0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. On the circle, these are heights.

Give the cosines of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ in order.

$1, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0$. On the circle, these are horizontal coordinates.

How do you rebuild tangent, cotangent, secant, and cosecant from sine and cosine?

$\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$. Each denominator must be nonzero.

State the Pythagorean trig identity and its two quotient forms.

$\sin^2 x + \cos^2 x = 1$. Divide by $\cos^2 x$ for $1 + \tan^2 x = \sec^2 x$, or by $\sin^2 x$ for $1 + \cot^2 x = \csc^2 x$, where defined.

How can you rewrite a product of sine and cosine of the same angle? What is the cosine double-angle identity?

$2 \sin x \cos x = \sin(2x)$ and $\cos(2x) = \cos^2 x - \sin^2 x$.

FORMULA REFERENCE / 2

Geometry for calculus

Draw the figure, label each variable, and say which lengths can change. For related rates, write the geometric equation before differentiating.

For a circle of radius r, give circumference and area.

$C = 2\pi r$ and $A = \pi r^2$. Diameter $d = 2r$.

For a right triangle, how do you relate lengths? How do you find the area of any triangle?

$a^2 + b^2 = c^2$, where c is the hypotenuse. For any triangle, $A = \frac{1}{2}bh$, with h perpendicular to the base.

A cylindrical tank has radius r and height h. Give volume and closed surface area.

$V = \pi r^2 h$. Two disks and the side give $S = 2\pi r^2 + 2\pi r h$. Omit one disk for an open top.

What is a cone's volume? In a cone with its tip at the bottom, how do water radius and depth relate?

$V = \frac{1}{3}\pi r^2 h$. For water depth $h > 0$ above the tip, $r/h = R/H$ by similar triangles. Full-cone radius and height are R, H.

For a sphere of radius r, give volume and surface area.

$V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$.

An interval of length w is the side of a square or equilateral triangle, or the diameter of a semicircle. Give each cross-sectional area.

Square w^2. Equilateral triangle $\frac{\sqrt{3}}{4}w^2$. Semicircle $\frac{\pi}{8}w^2$ since its radius is $\frac{w}{2}$.

What area do you integrate for a disk or washer perpendicular to the axis of rotation?

$A = \pi\left(R^2 - r^2\right)$, with R and r the outer and inner distances to the axis. A disk has $r = 0$.

FORMULA REFERENCE / 3

Differentiation rules

Read each expression from the outside in. Name the rule and point to its factors or inner function before writing a derivative.

What are the constant and power rules? How do you differentiate a sum?

$\frac{d}{dx}c = 0$ and $\frac{d}{dx}x^n = nx^{n-1}$ where x^n is real and differentiable. For example, $\left(x^{-2}\right)' = -2x^{-3}$ requires $x \neq 0$. Differentiate sums term by term.

Give the product and quotient rules for differentiable u, v.

$(uv)' = u'v + uv'$. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ where $v \neq 0$.

Give the chain rule for a composite function.

If $y = f\left(g(x)\right)$, then $y' = f'\left(g(x)\right)g'(x)$, provided both derivatives exist at the relevant inputs.

Differentiate e^u and a^u when $u = u(x)$.

$(e^u)' = e^u u'$. $(a^u)' = a^u \ln(a)\, u'$ for constant $a > 0$.

Differentiate $\ln|u|$ and $\log_a u$ when $u = u(x)$.

$(\ln|u|)' = \frac{u'}{u}$ for $u \neq 0$. $\left(\log_a u\right)' = \frac{u'}{u \ln a}$ for $u > 0$, $a > 0$, $a \neq 1$.

How do you obtain an inverse function's derivative from a matching point on the original graph?

If $f(a) = b$ and f has a local differentiable inverse with $f'(a) \neq 0$, then $\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$.

Give the tangent-line approximation near $x = a$.

$L(x) = f(a) + f'(a)(x - a)$. Use $L(x)$ to estimate $f(x)$ close to a when f is differentiable there.

FORMULA REFERENCE / 4

Trig and inverse-trig derivatives

Use radians. Practice the basic forms first, then replace the angle by an inner function and include its derivative.

What are the sine and cosine derivatives?

Angles are in radians. $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

What are the tangent and cotangent derivatives?

Angles are in radians. $(\tan x)' = \sec^2 x$ and $(\cot x)' = -\csc^2 x$, wherever the original functions are defined.

What are the secant and cosecant derivatives?

Angles are in radians. $(\sec x)' = \sec x \tan x$ and $(\csc x)' = -\csc x \cot x$, wherever the functions are defined.

Differentiate $\arcsin u$ and $\arccos u$.

Inverse-trig outputs are in radians. $(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$ and $(\arccos u)' = -\frac{u'}{\sqrt{1-u^2}}$ for $|u| < 1$.

Differentiate $\arctan u$.

Inverse-trig outputs are in radians. $(\arctan u)' = \frac{u'}{1+u^2}$ for every real u where u' exists.

FORMULA REFERENCE / 5

Common antiderivatives

Identify the derivative pattern you see. Give the constant of integration and check your result by taking its derivative.

Find $\int x^n\,dx$. Which exponent requires a different rule?

For $n \neq -1$, use $\frac{x^{n+1}}{n+1} + C$ on an interval where the powers are defined. For $n = -1$, use $\ln|x| + C$ on an interval avoiding zero.

Find $\int e^x\,dx$ and $\int a^x\,dx$.

$e^x + C$ and $\frac{a^x}{\ln a} + C$, respectively, for $a > 0$, $a \neq 1$ in the second formula.

Find $\int \sin x\,dx$ and $\int \cos x\,dx$.

Angles are in radians. $-\cos x + C$ and $\sin x + C$, respectively.

Find $\int \sec^2 x\,dx$ and $\int \csc^2 x\,dx$.

Angles are in radians. $\tan x + C$ and $-\cot x + C$, respectively, on intervals where the integrands are defined.

Find $\int \sec x \tan x\,dx$ and $\int \csc x \cot x\,dx$.

Angles are in radians. $\sec x + C$ and $-\csc x + C$, respectively, on intervals where the integrands are defined.

Find $\int \frac{1}{1+x^2}\,dx$ and $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Inverse-trig outputs are in radians. $\arctan x + C$ for all real x and $\arcsin x + C$ for $|x| < 1$, respectively.

Find $\int \tan x\,dx$ and $\int \cot x\,dx$.

Angles are in radians. $-\ln|\cos x| + C$ and $\ln|\sin x| + C$, respectively, on intervals where the integrands are defined.

FORMULA REFERENCE / 6

Accumulation and substitution

Name the input, bounds, and units. With a variable endpoint, check whether differentiating also calls for the chain rule.

How do you evaluate a definite integral using an antiderivative?

If f is continuous on $[a, b]$ and $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

If $H(x) = \int_a^{g(x)} f(t)\,dt$, what is $H'(x)$?

$H'(x) = f\big(g(x)\big)g'(x)$ for continuous f and differentiable g. If $g(x)$ is the lower endpoint and the upper endpoint is fixed, reverse the sign.

How do the bounds change under the substitution $u = g(x)$?

With $du = g'(x)\,dx$, use $u = g(a)$ and $u = g(b)$ as the new bounds. Thus $\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ for continuous f and continuously differentiable g.

How do you distinguish average value from average rate of change?

On $[a, b]$ with $a < b$, average value is $\frac{1}{b-a}\int_a^b f(x)\,dx$ and has the units of f. Average rate of change is $\frac{f(b)-f(a)}{b-a}$ and has units of f divided by units of x.

How do displacement and total distance come from velocity?

On $a \le t \le b$, displacement is $\int_a^b v(t)\,dt$. Distance is $\int_a^b |v(t)|\,dt$. Split the interval where velocity changes sign if integrating by hand.

FORMULA REFERENCE / 7

Theorems and their conditions

For every theorem, recall the hypotheses as well as the conclusion. Practice writing one sentence that connects a condition to the given function.

When does the Intermediate Value Theorem guarantee a solution to $f(x) = k$?

If f is continuous on $[a, b]$ and k lies strictly between $f(a)$ and $f(b)$, then some c in (a, b) has $f(c) = k$.

When are both absolute extrema guaranteed? Where do you look for their values?

Continuous f on closed $[a, b]$ attains both. Compare f at each endpoint and at interior critical numbers, where $f' = 0$ or f' fails to exist.

State the conditions and conclusion of the Mean Value Theorem.

If f is continuous on $[a, b]$ and differentiable on (a, b), some c in (a, b) satisfies $f'(c) = \frac{f(b) - f(a)}{b - a}$.

When can you apply L'Hôpital's rule to a quotient limit?

First verify a form $0/0$ or ∞/∞. On the approach interval, both functions must be differentiable with $g' \neq 0$. If $\lim \frac{f'}{g'}$ exists (finite or infinite), it gives $\lim \frac{f}{g}$.

FRAMEWORK AND FURTHER READING

Sources

The examples and exercises were developed for this home program, with the two OpenStax adaptations credited below. The topic numbers identify the corresponding skills in the College Board framework. OpenStax section numbers point to additional explanations, graphs, and exercises.

AP Calculus AB and BC Course and Exam Description

College Board. Use the AB topics in Units 1–8.

https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf

Fall 2026 clarifications and corrections

College Board. Read alongside the course framework.

https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-bc-course-and-exam-description-clarifications.pdf

Calculus Volume 1

Gilbert Strang and Edwin “Jed” Herman, OpenStax. Functions, limits, derivatives, integration, and applications.

https://openstax.org/details/books/calculus-volume-1

Calculus Volume 2

Gilbert Strang and Edwin “Jed” Herman, OpenStax. Refer to the specified readings on numerical integration and differential equations.

https://openstax.org/details/books/calculus-volume-2

OpenStax adaptations

Gilbert Strang, Edwin “Jed” Herman, and contributors, Calculus Volume 1 (OpenStax). Topic 2.3 adapts §3.1, Example 3.4. Topic 6.4 adapts §5.3, Example 5.18. Both add explanations and a new instructional emphasis. The adapted material is licensed under CC BY-NC-SA 4.0.

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